# WEAKLY $(I, J)$-CONTINUOUS MULTIFUNCTIONS AND CONTRA $(I, J)$-CONTINUOUS MULTIFUNCTIONS

**E. Rosas**[*]
*Departamento de Matemáticas*
*Universidad de Oriente*
*Cumaná, Venezuela*
*and*
*Departamento de Ciencias Naturales y Exactas*
*Universidad de la Costa,*
*Barranquilla, Colombia*
*ennisrafael@gmail.com*
*erosas@cuc.edu.co*

**C. Carpintero**
*Departamento de Matemáticas*
*Universidad De Oriente*
*Cumaná, Venezuela*
*and*
*Universidad Autónoma del Caribe*
*Barranquilla, Colombia*
*carpintero.carlos@gmail.com*

**J. Sanabria**
*Facultad de Ciencias Básicas*
*Universidad del Atántico*
*Barranquilla, Colombia*
*jesanabri@gmail.com*

**Abstract.** The purpose of the present paper is to introduce, study and characterize upper and lower weakly $(I, J)$-continuous multifunctions and contra $(I, J)$-continuous multifunctions. Also, we investigate its relation with another class of continuous multifunctions.
**Keywords:** weakly $(I, J)$-continuous multifunctions, $I$-open set, $I$-closed set, contra $(I, J)$-continuous multifunctions, $(I, J)$-continuous multifunctions.

## 1. Introduction

It is well known today, that the notion of multifunction playing a very important role in general topology, upper and lower continuity have been extensively studied on multifunctions $F : (X, \tau) \to (Y, \sigma)$. Currently using the notion of ideal topological space, different types of upper and lower continuity in a multifunction $F : (X, \tau, I) \to (Y, \sigma)$ have been studied and characterized [2], [8], [9], [15],

∗. Corresponding author

[18]. The concept of ideal topological spaces has been introduced and studied by Kuratowski [12] and the local function of a subset $A$ of a topological space $(X, \tau)$ was introduced by Vaidyanathaswamy [17] as follows: given a topological space $(X, \tau)$ with an ideal $I$ on $X$ and if $P(X)$ is the set of all subsets of $X$, a set operator $(.)^* : P(X) \to P(X)$, called the local function of $A$ with respect to $\tau$ and $I$, is defined as follows: for $A \subseteq X$, $A^*(\tau, I) = \{x \in X \ / \ U \cap A \notin I$ for every $U \in \tau_x\}$, where $\tau_x = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*(,)$ for a topology $\tau^*(\tau, I)$ called the *-topology, finer than $\tau$ is defined by $cl^*(A) = A \cup A^*(\tau, I)$. We will denote $A^*(\tau, I)$ by $A^*$. In 1990, Jankovic and Hamlett[10], introduced the notion of $I$-open set in a topological space $(X, \tau)$ with an ideal $I$ on $X$. In 1992, Abd El-Monsef et al.[1] further investigated $I$-open sets and $I$-continuous functions. In 2007, Akdag [2], introduced the concept of $I$-continuous multifunctions in a topological space with and ideal on it. In 2007, Al-Omari and Noorani [3], introduced the notions of contra-$I$-continuous and almost $I$-continuous functions. Given a multifunction $F : (X, \tau) \to (Y, \sigma)$, and two ideals $I, J$ associate, now with the topological spaces $(X, \tau, I)$ and $(Y, \sigma, J)$, consider the multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$. We want to study some type of upper and lower continuity of $F$ as doing Rosas et al. [14]. In this paper, we introduce and study two new classes of multifunctions called a weakly $(I, J)$-continuous multifunctions and contra $(I, J)$-continuous multifunctions in topological spaces. Investigate its relation with another classes of continuous multifunctions. Also its relation when the ideal $J = \{\emptyset\}$.

## 2. Preliminaries

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) always mean topological spaces in which no separation axioms are assumed, unless explicitly stated and if $I$ is and ideal on $X$, $(X, \tau, I)$ mean an ideal topological space. For a subset $A$ of $(X, \tau)$, $Cl(A)$ and $\text{int}(A)$ denote the closure of $A$ with respect to $\tau$ and the interior of $A$ with respect to $\tau$, respectively. A subset $A$ is said to be regular open [16] (resp. semiopen [11], preopen[13], semi preopen [4]) if $A = \text{int}(Cl(A))$ (resp.$A \subseteq Cl(\text{int}(A)), A \subseteq \text{int}(Cl(A)), A \subseteq Cl(\text{int}(Cl(A)))$). The complement of a regular open (resp. semiopen, semi-preopen) set is called regular closed (resp. semiclosed, semi-preclosed) set. A subset $S$ of $(X, \tau, I)$ is an $I$-open[10], if $S \subseteq \text{int}(S^*)$. The complement of an $I$-open set is called $I$-closed set. The $I$-closure and the $I$-interior, can be defined in the same way as $Cl(A)$ and $\text{int}(A)$, respectively, will be denoted by $I\,Cl(A)$ and $I\text{int}(A)$, respectively. The family of all $I$-open (resp. $I$-closed, regular open, regular closed, semiopen, semi closed, preopen, semi-preclosed) subsets of a $(X, \tau, I)$, denoted by $IO(X)$(resp. $IC(X), RO(X), RC(X), SO(X), SC(X), PO(X), SPO(X), SPC(X)$). We set $IO(X, x) = \{A : A \in IO(X)$ and $x \in A\}$. It is well known that in an ideal topological space $(X, \tau, I)$, $X^* \subseteq X$ but if the ideal is codense, that is $\tau \cap I = \emptyset$, then $X^* = X$.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from $X$ into $Y$, also we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, the upper and lower inverse of any subset $A$ of $Y$ denoted by $F^+(A)$ and $F^-(A)$, respectively, that is $F^+(A) = \{x \in X : F(x) \subseteq A\}$ and $F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}$. In particular, $F^+(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$.

**Definition 2.1** ([7]). *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

1. *upper semi continuous at a point $x \in X$ if for each open set $V$ of $Y$ with $x \in F^+(V)$, there exists an open set $U$ containing $x$ such that $F(U) \subseteq V$.*
2. *lower semi continuous at a point $x \in X$ if for each open set $V$ of $Y$ with $F(x) \cap V \neq \emptyset$, there exists an open set $U$ containing $x$ such that $F(a) \cap V \neq \emptyset$ for all $a \in U$.*

**Definition 2.2** ([15]). *A multifunction $F : (X, \tau) \to (Y, \sigma)$ is said to be:*

1. *upper weakly continuous if for each $x \in X$ and each open set $V$ of $Y$ such that $x \in F^+(V)$, there exists an open set $U$ containing $x$ such that $U \subseteq F^+(Cl(V))$.*
2. *lower weakly continuous if for each $x \in X$ and each open set $V$ of $Y$ such that $F(x) \cap V \neq \emptyset$, there exists an open set $U$ containing $x$ such that $F(u) \cap Cl(V) \neq \emptyset$ for every $u \in U$.*
3. *weakly continuous if it is both upper weakly continuous and lower weakly continuous.*

**Definition 2.3** ([2]). *A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be:*

1. *upper $I$-continuous if for each $x \in X$ and each open set $V$ of $Y$ such that $x \in F^+(V)$, there exists an $I$-open set $U$ containing $x$ such that $U \subseteq F^+(V)$.*
2. *lower $I$-continuous if for each $x \in X$ and each open set $V$ of $Y$ such that $x \in F^-(V)$, there exists an $I$-open set $U$ containing $x$ such that $U \subseteq F^-(V)$.*
3. *$I$-continuous if it is both upper and lower $I$-continuous.*

**Definition 2.4** ([5]). *A multifunction $F : (X, \tau, I) \to (Y, \sigma)$ is said to be:*

1. *upper weakly $I$-continuous if for each $x \in X$ and each open set $V$ of $Y$ such that $x \in F^+(V)$, there exists an $I$-open set $U$ containing $x$ such that $U \subseteq F^+(Cl(V))$.*
2. *lower weakly $I$-continuous if for each $x \in X$ and each open set $V$ of $Y$ such that $x \in F^-(V)$, there exists an $I$-open set $U$ containing $x$ such that $U \subseteq F^-(Cl(V))$*
3. *weakly $I$-continuous if it is both upper weakly $I$-continuous and lower $I$-weakly continuous.*

## 3. Weakly $(I, J)$-continuous multifunctions

**Definition 3.1.** *A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *upper weakly $(I, J)$-continuous at a point $x \in X$ if for each $J$-open set $V$ such that $x \in F^+(V)$, there exists an $I$-open set $U$ containing $x$ such that $U \subseteq F^+(J\,Cl(V))$*
2. *lower weakly $(I, J)$-continuous at a point $x \in X$ if for each $J$-open set $V$ of $Y$ such that $x \in F^-(V)$, there exists an $I$-open set $U$ of $X$ containing $x$ such that $U \subseteq F^-(J\,Cl(V))$.*
3. *upper (resp. lower) weakly $(I, J)$-continuous on $X$ if it has this property at every point of $X$.*

**Example 3.2.** Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b\}\}$ $\sigma = \{\emptyset, Y, \{a\}\}$ and two ideals $I = \{\emptyset, \{a\}\}$, $J = \{\emptyset, \{b\}\}$. Define a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(a) = \{a\}$, $F(b) = \{c\}$ and $F(c) = \{b\}$. It is easy to see that:

The set of all $I$-open is $\{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

The set of all $J$-open is $\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, Y\}$.

In consequence, $F$ is upper(resp. lower) weakly $(I, J)$-continuous on $X$.

**Example 3.3.** Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{b\}\}$ and two ideals $I = J = \{\emptyset, \{b\}\}$. Define a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(a) = \{a\}$, $F(b) = \{c\}$ and $F(c) = \{b\}$. It is easy to see that:

The set of all $I$-open is $\{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$.

The set of all $J$-open is $\{\emptyset, Y, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. In consequence, $F$ is not upper (resp. lower) weakly $(I, J)$-continuous.

Recall that if $(X, \tau, I)$ is an ideal topological space and $I$ is the empty ideal, then for each $A \subseteq X$, $A^* = cl(A)$, that is to said, every $I$-open set is a pre-open set, in consequence, if $F : (X, \tau, I) \to (Y, \sigma, \{\emptyset\})$ is upper weakly $(I, \{\emptyset\})$-continuous, then $F$ is upper weakly $I$-continuous.

**Example 3.4.** Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b\}\}$ $\sigma = \{\emptyset, Y, \{a, c\}\}$ and two ideals $I = \{\emptyset, \{a\}\}$, $J = \{\emptyset\}$. Define a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(a) = \{b\}$, $F(b) = \{c\}$ and $F(c) = \{a\}$. It is easy to see that:

The set of all $I$-open is $\{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

The set of all $J$-open is $\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, Y\}$.

$F : (X, \tau, I) \to (Y, \sigma)$ is upper weakly $I$-continuous but $F : (X, \tau, I) \to (Y, \sigma, \{\emptyset\})$ is not upper weakly $(I, \{\emptyset\})$-continuous.

Now consider $(X, \tau, I)$ and $(Y, \sigma, J)$ two ideals topological spaces. If $J \neq \{\emptyset\}$, then the concepts of upper weakly $(I, J)$-continuous and upper weakly $I$-continuous are independent, as we can see in the following examples.

**Example 3.5.** In the Example 3.4, the multifunction $F$ is upper weakly $(I, J)$-continuous on $X$ but is not upper weakly $I$-continuous on $X$.

**Example 3.6.** In the Example 3.3, the multifunction $F$ is upper weakly $I$-continuous on $X$ but is not upper weakly $(I, J)$-continuous on $X$.

**Remark 3.7.** It is easy to see that if $F : (X, \tau, I) \to (Y, \sigma, J)$ is a multifunction and $JO(Y) \subset \sigma$ and $F$ is upper (lower) weakly $I$-continuous, then $F$ is upper (lower) weakly $(I, J)$-continuous. Even more, if $F : (X, \tau, I) \to (Y, \sigma, J)$ is a multifunction and $JO(Y) \nsubseteq \sigma$, we can find upper (resp. lower) weakly $(I, J)$-continuous on $X$ that are not upper (lower) weakly $I$-continuous.

The following theorem characterize the upper weakly $(I, J)$ continuous multifunctions.

**Theorem 3.8.** *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *$F$ is upper weakly $(I, J)$-continuous.*
2. *$F^+(V) \subseteq I\,int(F^+(J\,Cl(V)))$ for any $J$-open set $V$ of $Y$.*
3. *$I\,Cl(F^-(J\,int(B))) \subset F^-(B)$ for any every $J$-closed subset $B$ of $Y$.*

**Proof.** (1)⇒(2): Let $x \in F^+(V)$ and $V$ be any $J$- open set of $Y$. From (1), there exists an $I$-open set $U_x$ containing $x$ such that $U_x \subset F^+(J\,Cl(V))$. It follows that $x \in I\text{int}(F^+(J\,Cl(V)))$, in consequence, $F^+(V) \subseteq I\text{int}(F^+(J\,Cl(V)))$ for any $J$-open set $V$ of $Y$.

(2)⇒(1): Let $V$ any $J$-open subset of $Y$ such that $x \in F^+(V)$. By (2), $x \in F^+(V) \subseteq I\text{int}(F^+(J\,Cl(V))) \subseteq F^+(J\,Cl(V))$. Choose $U = I\text{int}(F^+(J\,Cl(V)))$. $U$ is an $I$-open subset of $X$, containing $x$. It follows that $F$ is upper weakly $(I, J)$-continuous.

(2)⇒(3): Let $B$ be any $J$- closed set of $Y$.

Then by (2), $F^+(Y \backslash B) = X \backslash F^-(B) \subseteq I\text{int}(F^+(J\,Cl(Y \backslash B)))$
$= I\text{int}(F^+(J\,Cl(Y \backslash I\text{int}(B)))) = X \backslash I\,Cl(F^-(J\text{int}(B)))$.
Thus, $I\,Cl(F^-(J\text{int}(B))) \subset F^-(B)$.

(3)⇒(2): Let $V$ be any $J$- open set of $Y$. Then by (3), $I\,Cl(F^-(J\text{int}(Y \backslash V))) \subset F^-(Y \backslash V) = X \backslash F^+(V)$.

It follows that $I\,Cl(X \backslash F^+(I\,Cl(V)) = I\,Cl(F^-(Y \backslash I\,Cl(V)))$
$= I\,Cl(F^-(J\text{int}(Y \backslash V))) \subset X \backslash F^+(V)$, and then $X \backslash I\text{int}(F^+(I\,Cl(V)))$
$\subseteq X \backslash F^+(V)$. Therefore the result follows. □

**Theorem 3.9.** *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *$F$ is lower weakly $(I, J)$-continuous.*
2. *$F^-(V) \subseteq I\,int(F^-(J\,Cl(V)))$ for any $J$-open set $V$ of $Y$.*

3. *$I\ Cl(F^+(J int(B))) \subset F^+(B)$ for any every $J$-closed subset $B$ of $Y$.*

**Proof.** The proof is similar to that of Theorem 3.8. □

**Definition 3.10** ([14]). *A multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *upper $(I, J)$-continuous at a point $x \in X$ if for each $J$-open set $V$ containing $F(x)$, there exists an $I$-open set $U$ containing $x$ such that $F(U) \subset V$.*

2. *lower $(I, J)$-continuous at a point $x \in X$ if for each $J$-open set $V$ of $Y$ meeting $F(x)$, there exists an $I$-open set $U$ of $X$ containing $x$ such that $F(u) \cap V \neq \emptyset$ for each $u \in U$.*

3. *upper (resp. lower) $(I, J)$-continuous on $X$ if it has this property at every point of $X$.*

**Example 3.11.** The multifunction defined in Example 3.2 is upper weakly $(I, J)$-continuous on $X$ but is not upper $(I, J)$-continuous on $X$.

**Remark 3.12.** Every upper (resp. lower) $(I, J)$-continuous multifunction on $X$ is upper (resp. lower) weakly $(I, J)$-continuous multifunction on $X$, but the converse is not necessarily true, as we can see in the following example.

**Example 3.13.** Let $X = \mathbb{R}$ the set of real numbers with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$, $Y = \mathbb{R}$ with the topology $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $I = \{\emptyset\}$=J. Define $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(x) = \mathbb{Q}$ if $x \in \mathbb{Q}$ and $F(x) = \mathbb{R} \setminus \mathbb{Q}$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Recall that in this case the $I$-open sets are the preopen sets. $f$ is upper (resp. lower) weakly $(I, J)$-continuous on $X$, but is not upper (resp. lower) $(I, J)$-continuous on $X$..

**Theorem 3.14** ([14]). *For a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$, the following statements are equivalent:*

1. *$F$ is upper $(I, J)$-continuous.*

2. *$F^+(V)$ is $I$-open for each $J$-open set $V$ of $Y$.*

3. *$F^-(K)$ is $I$-closed for every $J$-closed subset $K$ of $Y$.*

4. *$I\ Cl(F^-(B)) \subset F^-(J\ Cl(B))$ for every subset $B$ of $Y$.*

5. *For each point $x \in X$ and each $J$-open set $V$ containing $F(x)$, $F^+(V)$ is an $I$-open containing $x$.*

There exist any additional condition in order to say that every upper (resp. lower) $(I, J)$-continuous if upper (resp. lower) weakly $(I, J)$-continuous.

**Theorem 3.15.** *Let $F : (X, \tau, I) \to (Y, \sigma, J)$ be a multifunction such that $F(x)$ is a $J$-open subset of $Y$ for each $x \in X$. Then $F$ is lower $(I, J)$-continuous if and only if lower weakly $(I, J)$-continuous.*

**Proof.** Let $x \in X$ and $V$ any $J$-open subset of $Y$ such that $x \in F^{-}(V)$. Then there exists an $I$-open subset $U$ of $X$ containing $x$ such that $U \subset F^{-}(J\,Cl(V)$. It follows that $F(u) \cap J\,Cl(V) \neq \emptyset$ for each $u \in U$. Since $F(u)$ is a $J$-open subset of $Y$ for each $u \in U$, it follows that $F(u) \cap V \neq \emptyset$ and then $F$ is lower $(I, J)$-continuous. The converse is clear because every $(I, J)$-continuous multifunction is weakly $(I, J)$-continuous. □

**Theorem 3.16.** *Let $F : (X, \tau, I) \to (Y, \sigma, J)$ be a multifunction such that $F(x)$ is a $J$-open subset of $Y$ for each $x \in X$. Then $F$ is upper $(I, J)$-continuous if and only if upper weakly $(I, J)$-continuous.*

**Proof.** The proof is similar to the above Theorem. □

## 4. Contra $(I, J)$-continuous multifunctions

**Definition 4.1.** *A multifunction $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *upper contra $(I, J)$-continuous if for each $x \in X$ and each $J$-closed set $V$ such that $x \in F^{+}(V)$, there exists an $I$-open set $U$ containing $x$ such that $F(U) \subset V$.*

2. *lower contra $(I, J)$-continuous if for each $x \in X$ and each $J$-closed set $V$ of $Y$ such that $x \in F^{-}(V)$, there exists an $I$-open set $U$ of $X$ containing $x$ such that $U \subseteq F^{-}(V)$.*

3. *Contra $(I, J)$-continuous if it is upper contra $(I, J)$-continuous and lower contra $(I, J)$-continuous.*

**Example 4.2.** Let $X = \mathbb{R}$ the set of real numbers with the topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}\}$, $Y = \mathbb{R}$ with the topology $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $I = \{\emptyset\}$=J. Define $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(x) = \mathbb{Q}$ if $x \in \mathbb{Q}$ and $F(x) = \mathbb{R} \setminus \mathbb{Q}$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Recall that in this case the $I$-open sets are the preopen sets. It is easy to see that $F$ is upper (resp. lower) contra $(I, J)$-continuous.

**Example 4.3.** Let $X = Y = \{a, b, c\}$ with two topologies $\tau = \{\emptyset, X, \{b\}\}$ $\sigma = \{\emptyset, Y, \{a\}\}$ and two ideals $I = \{\emptyset, \{a\}\}$, $J = \{\emptyset, \{b\}\}$. Define a multifunction $F : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $F(a) = \{b\}$, $F(b) = \{a\}$ and $F(c) = \{c\}$. It is easy to see that:

The set of all $I$-open is $\{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

The set of all $J$-open is $\{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, Y\}$.

The set of all $J$-closed is $\{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, Y\}$.

In consequence, $f$ is upper(resp. lower) $(I, J)$-continuous on $X$ but is not upper (resp. lower) contra $(I, J)$-continuous.

**Example 4.4.** The multifunction $F$ defined in Example 4.2 is upper (resp. lower) contra $(I, J)$-continuous but is not upper (resp. lower) $(I, J)$-continuous on $X$ and the multifunction $F$ defined in Example 4.3 is upper (resp. lower)

$(I,J)$-continuous but is not upper (resp. lower) contra $(I,J)$-continuous. In consequence both concepts are independent of each other.

**Theorem 4.5.** *For a multifunction $F:(X,\tau,I)\to(Y,\sigma,J)$, the following statements are equivalent:*

1. *$F$ is upper contra $(I,J)$-continuous.*
2. *$F^+(V)$ is $I$-open for each $J$-closed set $V$ of $Y$.*
3. *$F^-(K)$ is $I$-closed for every $J$-open subset $K$ of $Y$.*

**Proof.** (1)⇔(2): Let $x\in F^+(V)$ and $V$ be any $J$-closed set of $Y$. From (1), there exists an $I$-open set $U_x$ containing $x$ such that $U_x\subset F^+(V)$. It follows that $F^+(V)=\bigcup_{x\in F^+(V)} U_x$. Since any union of $I$-open sets is $I$-open, $F^+(V)$ is $I$-open in $(X,\tau)$. The converse is similar.

(2)⇔(3): Let $K$ be any $J$- open set of $Y$. Then $Y\backslash K$ is a $J$-closed set of $Y$ by (2), $F^+(Y\backslash K)=X\backslash F^-(K)$ is an $I$-open set. Then it is obtained that $F^-(K)$ is an $I$-closed set. The converse is similar. □

**Theorem 4.6.** *For a multifunction $F:(X,\tau,I)\to(Y,\sigma,J)$, the following statements are equivalent:*

1. *$F$ is lower contra $(I,J)$-continuous.*
2. *$F^-(V)$ is $I$-open for each $J$-closed set $V$ of $Y$.*
3. *$F^+(K)$ is $I$-closed for every $J$-open subset $K$ of $Y$.*
4. *For each $x\in X$ and each $J$-closed set $K$ of $Y$ such that $F(x)\cap K\neq\emptyset$, there exists an $I$-open set $U$ containing $x$ such that $F(y)\cap K\neq\emptyset$ for each $y\in U$.*

**Proof.** The proof is similar to the proof of Theorem 4.5. □

**Remark 4.7.** It is easy to see that if $J=\{\emptyset\}$ and $F:(X,\tau,I)\to(Y,\sigma,J)$ is upper (resp. lower) contra $(I,J)$-continuous then $F$ is upper (resp. lower) contra $I$-continuous.

The following example shows the existence of upper (resp. lower) contra $I$-continuous that is not upper (resp. lower) contra $(I,\{\emptyset\})$-continuous.

**Example 4.8.** Let $X=Y=\{a,b,c\}$ with two topologies $\tau=\{\emptyset,X,\{b\}\}$ $\sigma=\{\emptyset,Y,\{a,c\}\}$ and two ideals $I=\{\emptyset,\{a\}\}$, $J=\{\emptyset\}$. Define a multifunction $F:(X,\tau,I)\to(Y,\sigma,J)$ as follows: $F(a)=\{c\}$, $F(b)=\{b\}$ and $F(c)=\{a\}$. It is easy to see that:

The set of all $I$-open is $\{\emptyset,X,\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$.

The set of all $J$-open is $\{\emptyset,\{a\},\{c\},\{a,b\},\{a,c\},\{b,c\},Y\}$.

The set of all $J$-closed is $\{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, Y\}$.

Observe that $F : (X, \tau, I) \to (Y, \sigma)$ is upper contra $I$-continuous but $F : (X, \tau, I) \to (Y, \sigma, \{\emptyset\})$ is not upper contra $(I, \{\emptyset\})$-continuous.

**Remark 4.9.** It is easy to see that if $F : (X, \tau, I) \to (Y, \sigma, J)$ is a multifunction and $JO(Y) \subset \sigma$. If $F$ is upper (lower) contra $I$-continuous, then $F$ is upper (lower) $(I, J)$-continuous. Even more, if $F : (X, \tau, I) \to (Y, \sigma, J)$ is a multifunction and $JO(Y) \nsubseteq \sigma$, we can find upper (resp. lower) contra $(I, J)$-continuous on $X$ that are not upper (lower) contra $I$-continuous.

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