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MATHEMATICS

Topology

ON DECOMPOSITION OF BIOPERATION-CONTINUITY

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Abstract

In this paper, we introduce some new types of sets via bioperation and obtain a new decomposition of bioperation-continuity using this sets.

Key words: topological spaces, $\gamma \vee \gamma'$-open set

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1. Introduction. Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms, etc. by utilizing generalized open sets. Kasahara [1] defined the concept of an operation on topological spaces. Ogata and Maki [2] introduced the notion of $\tau_{\gamma\vee\gamma'}$ which is the collection of all $\gamma\vee\gamma'$-open sets in a topological space (X,τ) and Umehara in [3] introduced the notion of $\tau_{(\gamma,\gamma')}$ which is the collection of all (γ,γ')-open sets in a topological space (X,τ) that generalized the $\gamma \vee \gamma'$-open sets in a topological space (X,τ). In this paper, we introduce some types of sets via bioperation and obtain a new decomposition of bioperation-continuity using these new described sets.

2. Preiliminaries. The closure and the interior of a subset A of (X,τ) are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1 ([1]). Let (X,τ) be a topological space. An operation γ on the topology τ is a function from τ to the power set $\mathcal{P}(X)$ of X such that $V \subset V^\gamma$ for each $V \in \tau$, where V^γ denotes the value of τ at V. It is denoted by $\gamma\colon \tau \to \mathcal{P}(X)$.

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Definition 2.2 ([2]). A topological space (X,τ) equipped with two operations, say, γ and γ' defined on τ is called a bioperation-topological space, it is denoted by (X,τ,γ,γ').

Definition 2.3 ([2]). A subset A of a topological space (X,τ) is said to be $\gamma\vee\gamma'$-open set if for each $x\in A$ there exists an open neighbourhood U of x such that $U^{\gamma}\cup U^{\gamma'}\subset A$. The complement of $\gamma\vee\gamma'$-open set is called $\gamma\vee\gamma'$-closed. $\tau_{\gamma\vee\gamma'}$ denotes the set of all $\gamma\vee\gamma'$-open sets in (X,τ).

Definition 2.4 ([3]). A subset A of a topological space (X,τ) is said to be (γ,γ')-open set if for each $x\in A$ there exist open neighbourhoods U and V of x such that $U^{\gamma}\cup W^{\gamma'}\subset A$. The complement of (γ,γ')-open set is called (γ,γ')-closed. $\tau_{(\gamma,\gamma')}$ denotes the set of all (γ,γ')-open sets in (X,τ).

Remark 2.5. Observe that every $\gamma\vee\gamma'$-open set is (γ,γ')-open set, but the converse is not necessarily true.

Definition 2.6 ([3]). For a subset A of (X,τ), $\tau_{(\gamma,\gamma')}$-Cl(A) denotes the intersection of all (γ,γ')-closed sets containing A, that is, $\tau_{(\gamma,\gamma')}\text{-Cl}(A)=\cap\{F\colon A\subset F, X\setminus F\in\tau_{(\gamma,\gamma')}\}$.

Definition 2.7. Let A be any subset of X. The $\tau_{(\gamma,\gamma')}$-Int(A) is defined as $\tau_{(\gamma,\gamma')}\text{-Int}(A)=\cup\{U\colon U \text{ is a } (\gamma,\gamma')\text{-open set and } U\subset A\}$.

Definition 2.8. Let (X,τ) be a topological space and A be a subset of X and γ and γ' be operations on τ. Then A is said to be

1. (γ,γ')-α-open if $A\subset \text{Int}_{(\gamma,\gamma')}(\text{Cl}_{(\gamma,\gamma')}(\text{Int}_{(\gamma,\gamma')}(A)))$,
2. (γ,γ')-preopen if $A\subset \text{Int}_{(\gamma,\gamma')}(\text{Cl}_{(\gamma,\gamma')}(A))$,
3. (γ,γ')-semiopen [4] if $A\subset \text{Cl}_{(\gamma,\gamma')}(\text{Int}_{(\gamma,\gamma')}(A))$,
4. (γ,γ')-semipreopen (or (γ,γ')-β-open) if $A\subset \text{Cl}_{(\gamma,\gamma')}(\text{Int}_{(\gamma,\gamma')}(\text{Cl}_{(\gamma,\gamma')}(A)))$,
5. (γ,γ')-regular open [5] if $A=\text{Int}_{(\gamma,\gamma')}(\text{Cl}_{(\gamma,\gamma')}(A))$.

Remark 2.9. The union of all (γ,γ')-semipreopen sets contained in A is called the (γ,γ')-semipreinterior of A and is denoted by $sp\,\text{Int}_{(\gamma,\gamma')}(A)$. The complement of a (γ,γ')-semipreopen set is called a (γ,γ')-semipreclosed set. It is clear that $sp\,\text{Int}_{(\gamma,\gamma')}(A)=A\cap\text{Cl}_{(\gamma,\gamma')}(\text{Int}_{(\gamma,\gamma')}(\text{Cl}_{(\gamma,\gamma')}(A)))$.

Definition 2.10. Let (X,τ) and (Y,σ) be two topological spaces and let $\gamma,\gamma'\colon\tau\to\mathcal{P}(X)$ be operations on τ. A mapping $f\colon(X,\tau)\to(Y,\sigma)$ is said to be (γ,γ')-continuous (resp. (γ,γ')-α-continuous, (γ,γ')-precontinuous, (γ,γ')-semicontinuous, (γ,γ')-semiprecontinuous) if for each $x\in X$ and each open set V of Y containing $f(x)$ there exists a (γ,γ')-open set U containing x (resp. (γ,γ')-α-open set, (γ,γ')-preopen set, (γ,γ')-semiopen set, (γ,γ')-semipreopen set) such that $f(U)\subset V$.

3. Some subsets in topological spaces. Through this section, let (X,τ) and (Y,σ) be two topological spaces, and let $\gamma,\gamma'\colon\tau\to\mathcal{P}(X)$ be operations on τ.

Definition 3.1. A subset A of a topological space (X,τ) with the operations γ, γ' is called:

1. $\alpha^\star_{(\gamma,\gamma')}$-set if $\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(A)))=\mathrm{Int}_{(\gamma,\gamma')}(A)$,
2. $t_{(\gamma,\gamma')}$-set if $\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(A))=\mathrm{Int}_{(\gamma,\gamma')}(A)$,
3. $s_{(\gamma,\gamma')}$-set if $\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(A))=\mathrm{Int}_{(\gamma,\gamma')}(A)$,
4. $\beta^\star_{(\gamma,\gamma')}$-set if $\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(A)))=\mathrm{Int}_{(\gamma,\gamma')}(A)$.

Example 3.2. Let $X=\{a,b,c\}$ and $\tau=\{\varnothing,X,\{a\},\{c\},\{a,c\},\{a,b\}\}$. We define the operations $\gamma,\gamma'\colon\tau\to\mathcal{P}(X)$ as follows

$$A^\gamma=\begin{cases}A & \text{if } A=\{a\},\\ A\cup\{a,c\} & \text{if } A\neq\{a\}\end{cases},\qquad A^{\gamma'}=\begin{cases}int(cl(A)) & \text{if } A=\{a\},\\ X & \text{if } A\neq\{a\}\end{cases}.$$

Observe that:

1. $\alpha^\star_{\gamma\vee\gamma'}$-set $=\{\varnothing,X,\{b\},\{c\},\{b,c\}\}$.
2. $t_{\gamma\vee\gamma'}$-set $=\{\varnothing,X,\{b\},\{c\},\{b,c\}\}$.
3. $s_{\gamma\vee\gamma'}$-set $=\{\varnothing,X,\{b\},\{c\},\{b,c\}\}$.
4. $\beta^\star_{\gamma\vee\gamma'}$-set $=\{\varnothing,X,\{b\},\{c\},\{b,c\}\}$.

Proposition 3.3. *The following statements are equivalent for a subset A of a space (X,τ) with the operations γ, γ':*

1. *A is an $\alpha^\star_{\gamma\vee\gamma'}$-set.*
2. *A is a $\gamma\vee\gamma'$-semipreclosed set.*
3. $\mathrm{Int}_{\gamma\vee\gamma'}(A)$ *is a $\gamma\vee\gamma'$-regular open set.*

Proof. Straightforward. □

Proposition 3.4. *Let A be a subset of a space (X,τ) with the operations γ, γ'.*

1. *A $\gamma\vee\gamma'$-semiopen set A is a $t_{\gamma\vee\gamma'}$-set if and only if it is an $\alpha^\star_{\gamma\vee\gamma'}$-set.*
2. *A is $\gamma\vee\gamma'$-α-open and an $\alpha^\star_{\gamma\vee\gamma'}$-set if and only if it is $\gamma\vee\gamma'$-regular open.*

Proof. 1. Let A be a $\gamma \vee \gamma'$-semiopen and A an $\alpha^{\star}_{\gamma\vee\gamma'}$-set. Since A is $\gamma \vee \gamma'$-semiopen, $\mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(A)) = \mathrm{Cl}_{\gamma\vee\gamma'}(A)$ and $\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(A)) = \mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(A))) = \mathrm{Int}_{\gamma\vee\gamma'}(A)$. Therefore, A is a $t_{\gamma\vee\gamma'}$-set.

2. Let A be a $\gamma\vee\gamma'$-α-open set and an $\alpha^{\star}_{\gamma\vee\gamma'}$-set. Then $\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(A)) = A$ and hence $\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(A)) = \mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(A))) = A$.

The converse is obvious. □

Definition 3.5. A subset A of a topological space (X, τ) with the operations γ, γ' is called:

1. $C_{\gamma\vee\gamma'}$-set if $A = U \cap V$, where $U \in \tau_{\gamma\vee\gamma'}$ and V is an $\alpha^{\star}_{\gamma\vee\gamma'}$-set;
2. $B_{\gamma\vee\gamma'}$-set if $A = U \cap V$, where $U \in \tau_{\gamma\vee\gamma'}$ and V is a $t_{\gamma\vee\gamma'}$-set;
3. $S_{\gamma\vee\gamma'}$-set if $A = U \cap V$, where $U \in \tau_{\gamma\vee\gamma'}$ and V is a $s_{\gamma\vee\gamma'}$-set;
4. $\beta_{\gamma\vee\gamma'}$-set if $A = U \cap V$, where $U \in \tau_{\gamma\vee\gamma'}$ and V is a $\beta^{\star}_{\gamma\vee\gamma'}$-set;
5. $\beta^{\star\star}$-open set if $sp\,\mathrm{Int}_{\gamma\vee\gamma'}(A) = \mathrm{Int}_{\gamma\vee\gamma'}(A)$.

Example 3.6. Observe that in Example 3.2

1. $C_{\gamma\vee\gamma'}$-set $= \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$.
2. $B_{\gamma\vee\gamma'}$-set $= \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$.
3. $S_{\gamma\vee\gamma'}$-set $= \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$.
4. $\beta_{\gamma\vee\gamma'}$-set $= \{\varnothing, X, \{b\}, \{c\}, \{b, c\}\}$.
5. $\beta^{\star\star}$-open set $= \{\varnothing, X, \{a\}, \{b, c\}\}$.

Proposition 3.7. *Let (X, τ) be a topological space with the operations γ, γ' and A a subset of X. Then the following statements hold:*

1. *If A is a $t_{\gamma\vee\gamma'}$-set, then A is an $\alpha^{\star}_{\gamma\vee\gamma'}$-set.*
2. *If A is a $s_{\gamma\vee\gamma'}$-set, then A is an $\alpha^{\star}_{\gamma\vee\gamma'}$-set.*
3. *If A is a $\beta^{\star}_{\gamma\vee\gamma'}$-set, then A is both $t_{\gamma\vee\gamma'}$-set and $s_{\gamma\vee\gamma'}$-set.*
4. *$t_{\gamma\vee\gamma'}$-set and $s_{\gamma\vee\gamma'}$-set are independent.*

Proof. 1. Let A be a $t_{\gamma\vee\gamma'}$-set. Then $\tau_{\gamma\vee\gamma'}$-$\mathrm{Int}(\tau_{\gamma\vee\gamma'}$-$\mathrm{Cl}(A)) = \tau_{\gamma\vee\gamma'}$-$\mathrm{Int}(A) \supset \tau_{\gamma\vee\gamma'}$-$\mathrm{Int}(\tau_{\gamma\vee\gamma'}$-$\mathrm{Cl}(\tau_{\gamma}$-$\mathrm{Int}(A))) \supset \tau_{\gamma\vee\gamma'}$-$\mathrm{Int}(A)$ and hence $\tau_{\gamma\vee\gamma'}$-$\mathrm{Int}(\tau_{\gamma\vee\gamma'}$-$\mathrm{Cl}(\tau_{\gamma\vee\gamma'}$-$\mathrm{Int}(A))) = \tau_{\gamma\vee\gamma'}$-$\mathrm{Int}(A)$. Therefore, A is an $\alpha^{\star}_{\gamma\vee\gamma'}$-set. □

Remark 3.8. The converses of the statements in Proposition 3.7 are not true as one can see from the following examples.

Example 3.9. Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$. We define the operations $\gamma, \gamma' \colon \tau \to \mathcal{P}(X)$ as follows

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } A = \{a\}, \\ A \cup \{a, c\} & \text{if } A \neq \{a\}. \end{cases}$$

Then $\tau_{\gamma\vee\gamma'} = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}\}$. If we take $A = \{a\}$, then A is an $\alpha^{\star}_{\gamma\vee\gamma'}$-set and a $t_{\gamma\vee\gamma'}$-set, but it is not an $s_{\gamma\vee\gamma'}$-set and not a $\beta^{\star}_{\gamma\vee\gamma'}$-set.

Example 3.10. Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. We define the operators $\gamma, \gamma' \colon \tau \to \mathcal{P}(X)$ by $\gamma(A) = \mathrm{Cl}(A)$ and $\gamma'(A) = \mathrm{Int}(\mathrm{Cl}(A))$ for all $A \in \tau$. Then $\tau_{\gamma\vee\gamma'} = \{\varnothing, X\}$. If $A = \{b\}$, then it is an $\alpha^{\star}_{\gamma\vee\gamma'}$-set and an $s_{\gamma\vee\gamma'}$-set, but it is not a $t_{\gamma\vee\gamma'}$-set and not a $\beta^{\star}_{\gamma\vee\gamma'}$-set.

Proposition 3.11. *Let (X, τ) be a topological space with the operations γ, γ' and A a subset of X. Then the following statements hold:*

1. *If A is an $\alpha^{\star}_{\gamma\vee\gamma'}$-set, then it is a $C_{\gamma\vee\gamma'}$-set.*
2. *If A is a $t_{\gamma\vee\gamma'}$-set, then it is a $B_{\gamma\vee\gamma'}$-set.*
3. *If A is an $s_{\gamma\vee\gamma'}$-set, then it is an $S_{\gamma\vee\gamma'}$-set.*
4. *If A is a $\beta^{\star}_{\gamma\vee\gamma'}$-set, then it is a $\beta_{\gamma\vee\gamma'}$-set.*

Proof. 1. Let A be an $\alpha^{\star}_{\gamma\vee\gamma'}$-set. If we take $U = X \in \tau_{\gamma\vee\gamma'}$, then $A = U \cap A$ and, hence, A is a $C_{\gamma\vee\gamma'}$-set. The proofs of 2, 3, and 4 are similar. □

Remark 3.12. The converses of the statements in Proposition 3.11 are not true. In Example 3.9, $\{a, c\}$ is a $C_{\gamma\vee\gamma'}$-set (resp. $B_{\gamma\vee\gamma'}$-set, $S_{\gamma\vee\gamma'}$-set, $\beta_{\gamma\vee\gamma'}$-set), but it is not an $\alpha^{\star}_{\gamma\vee\gamma'}$-set (resp. $t_{\gamma\vee\gamma'}$-set, $s_{\gamma\vee\gamma'}$-set, $\beta^{\star}_{\gamma\vee\gamma'}$-set).

Proposition 3.13. *Let (X, τ) be a topological space with the operations γ, γ'.*

1. *Every $B_{\gamma\vee\gamma'}$-set is a $C_{\gamma\vee\gamma'}$-set.*
2. *Every $S_{\gamma\vee\gamma'}$-set is a $C_{\gamma\vee\gamma'}$-set.*
3. *Every $\beta_{\gamma\vee\gamma'}$-set is both a $B_{\gamma\vee\gamma'}$-set and an $S_{\gamma\vee\gamma'}$-set.*

Remark 3.14. The converses of the statements in Proposition 3.13 are not true and $B_{\gamma\vee\gamma'}$-set and $S_{\gamma\vee\gamma'}$-set are independent notions. In Example 3.9, $\{a, b\}$ is a $B_{\gamma\vee\gamma'}$-set but it is not an $S_{\gamma\vee\gamma'}$-set and not a $\beta_{\gamma\vee\gamma'}$-set. In Example 3.10, $\{b\}$ is a $C_{\gamma\vee\gamma'}$-set and an $S_{\gamma\vee\gamma'}$-set but it is not a $B_{\gamma\vee\gamma'}$-set and not a $\beta_{\gamma\vee\gamma'}$-set.

Proposition 3.15. *Let (X, τ) be a topological space with the operations γ, γ' and A a subset of X. Then $\beta^{\star\star}$-open set and $\beta_{\gamma\vee\gamma'}$-set are equivalent.*

Proof. Let A be a $\beta^{\star}_{\gamma\vee\gamma'}$-set. Then $\mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(A))) = \mathrm{Int}_{\gamma\vee\gamma'}(A)$. Hence A is $\beta_{\gamma\vee\gamma'}$-set. Hence $\beta\,\mathrm{Int}_{\gamma\vee\gamma'}(A) = A \cap \mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(A))) = A \cap \mathrm{Int}_{\gamma\vee\gamma'}(A) = \mathrm{Int}_{\gamma\vee\gamma'}(A)$. Thus A is a $\beta^{\star\star}$-open set. Conversely, let A be a $\beta^{\star\star}$-open set. Then $\beta\,\mathrm{Int}_{\gamma\vee\gamma'}(A) = \mathrm{Int}_{\gamma\vee\gamma'}(A)$. Hence $\beta\,\mathrm{Int}_{\gamma\vee\gamma'}(A)$ is a $\gamma\vee\gamma'$-open set. Since $A = A \cap X$, A is a $\beta_{\gamma\vee\gamma'}$-set. □

Remark 3.16. We have the following implication diagram.

$$
\begin{array}{ccccccc}
\beta_{\gamma\vee\gamma'}\text{-set} & & \rightarrow & \beta^{\star}_{\gamma\vee\gamma'}\text{-set} & \rightarrow & & \beta_{\gamma\vee\gamma'}\text{-set} \\
\downarrow & & \swarrow & & \searrow & & \downarrow \\
S_{\gamma\vee\gamma'}\text{-set} & \rightarrow & s_{\gamma\vee\gamma'}\text{-set} & \downarrow & t_{\gamma\vee\gamma'}\text{-set} & \rightarrow & B_{\gamma\vee\gamma'}\text{-set} \\
& \searrow & \searrow & & \swarrow & \swarrow & \\
& & & \alpha^{\star}_{\gamma\vee\gamma'}\text{-set} & & & \\
& & \searrow & \downarrow & \swarrow & & \\
& & & C_{\gamma\vee\gamma'}\text{-set} & & &
\end{array}
$$

Theorem 3.17. *For a subset A of a space (X, τ) with the operations γ, γ', the following properties are equivalent:*

(1) *A is $\gamma\vee\gamma'$-open.*

(2) *A is a $\gamma\vee\gamma'$-α-open set and a $C_{\gamma\vee\gamma'}$-set.*

(3) *A is a $\gamma\vee\gamma'$-preopen set and a $B_{\gamma\vee\gamma'}$-set.*

(4) *A is a $\gamma\vee\gamma'$-semiopen set and an $S_{\gamma\vee\gamma'}$-set.*

(5) *A is a $\gamma\vee\gamma'$-semipreopen set and a $\beta_{\gamma\vee\gamma'}$-set.*

Proof. The proof of (1) → (2), (1) → (3), (1) → (4), (1) → (5) are obvious.

(5) → (1): Let A be a $\gamma\vee\gamma'$-semipreopen set and a $\beta_{\gamma\vee\gamma'}$-set. Since A is $\beta_{\gamma\vee\gamma'}$-set, $A = U \cap V$, where U is a $\gamma\vee\gamma'$-open set and V is a $\beta^{\star}_{\gamma\vee\gamma'}$-set. By the hypothesis, A is also $\gamma\vee\gamma'$-semipreopen and we have $A \subset \mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(A))) = \mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(U \cap V))) \subset \mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(U) \cap \mathrm{Cl}_{\gamma\vee\gamma'}(V))) = \mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(U)) \cap \mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(V))) \subset \mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(U))) \cap \mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(V))) \subset \mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(Cl_{\gamma\vee\gamma'}(U))) \cap \mathrm{Int}_{\gamma\vee\gamma'}(V)$. Hence $A = U \cap V = (U \cap V) \cap U \subset (\mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(U))) \cap \mathrm{Int}_{\gamma\vee\gamma'}(V)) \cap U = (\mathrm{Cl}_{\gamma\vee\gamma'}(\mathrm{Int}_{\gamma\vee\gamma'}(\mathrm{Cl}_{\gamma\vee\gamma'}(U))) \cap U) \cap \mathrm{Int}_{\gamma\vee\gamma'}(V)$. Notice that $A = U \cap V \supset U \cap \mathrm{Int}_{\gamma\vee\gamma'}(V)$. Hence $A = U \cap \mathrm{Int}_{\gamma\vee\gamma'}(V)$.

(2) → (1), (3) → (1), (4) → (1) are shown similarly. □

4. Decompositions of $\gamma\vee\gamma'$-continuity.

Definition 4.1. A function $f\colon (X, \tau) \to (Y, \sigma)$ is said to be a $C_{\gamma\vee\gamma'}$-continuous (resp. $B_{\gamma\vee\gamma'}$-continuous, $S_{\gamma\vee\gamma'}$-continuous, $\beta_{\gamma\vee\gamma'}$-continuous). If for each $V \in \sigma$, $f^{-1}(V)$ is a $C_{\gamma\vee\gamma'}$-set (resp. $B_{\gamma\vee\gamma'}$-set, $S_{\gamma\vee\gamma'}$-set, $\beta_{\gamma\vee\gamma'}$-set).

Proposition 4.2. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be a function and $\gamma\colon \tau \to \mathcal{P}(X)$ and $\gamma'\colon \tau \to \mathcal{P}(X)$ be two operations on τ. Then*

1. *Every $B_{\gamma\vee\gamma'}$-continuous function is $C_{\gamma\vee\gamma'}$-continuous.*
2. *Every $S_{\gamma\vee\gamma'}$-continuous function is $C_{\gamma\vee\gamma'}$-continuous.*
3. *Every $\beta_{\gamma\vee\gamma'}$-continuous is both $B_{\gamma\vee\gamma'}$-continuous and $S_{\gamma\vee\gamma'}$-continuous.*

Proof. The proof follows from Proposition 3.13. □

Theorem 4.3. *Let (X,τ) and (Y,σ) be two topological spaces and let $\gamma \vee \gamma'\colon \tau \to \mathcal{P}(X)$ be two operations on τ. For a function $f\colon (X,\tau) \to (Y,\sigma)$, the following properties are equivalent*:

1. *f is $\gamma \vee \gamma'$-continuous.*
2. *f is $\gamma \vee \gamma'$-α-continuous and $C_{\gamma\vee\gamma'}$-continuous.*
3. *f is $\gamma \vee \gamma'$-precontinuous and $B_{\gamma\vee\gamma'}$-continuous.*
4. *f is $\gamma \vee \gamma'$-semicontinuous and $S_{\gamma\vee\gamma'}$-continuous.*
5. *f is $\gamma \vee \gamma'$-semiprecontinuous and $\beta_{\gamma\vee\gamma'}$-continuous.*

Proof. The proof follows from Theorem 3.17. □

Remark 4.4. The notions of $\gamma\vee\gamma'$-α-continuity and $C_{\gamma\vee\gamma'}$-continuity, $\gamma\vee\gamma'$-continuity and $B_{\gamma\vee\gamma'}$-continuity, $\gamma\vee\gamma'$-semicontinuity and $S_{\gamma\vee\gamma'}$-continuity, $\gamma\vee\gamma'$-semiprecontinuity and $\gamma \vee \gamma'$-continuity are independent of each other as seen in the following examples.

Example 4.5. Let $X = \{a,b,c\}$ and $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a,c\}, \{a,b\}\}$ and $\sigma = \{\varnothing, X, \{a\}, \{c\}, \{a,c\}\}$. We define the operators $\gamma, \gamma'\colon \tau \to \mathcal{P}(X)$ by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } A = \{a\}, \\ A \cup \{a,c\} & \text{if } A \neq \{a\}. \end{cases}$$

Then $\tau_{\gamma\vee\gamma'} = \{\varnothing, X, \{a\}, \{c\}, \{a,c\}\}$. Define a function $f\colon (X,\tau) \to (Y,\sigma)$ as $f(a) = f(b) = a$, $f(c) = c$. Then f is $C_{\gamma\vee\gamma'}$-continuous (resp. $B_{\gamma\vee\gamma'}$-continuous, $\gamma \vee \gamma'$-semicontinuous and $\gamma \vee \gamma'$-semiprecontinuous), but it is not $\gamma \vee \gamma'$-α-continuous (resp. $\gamma \vee \gamma'$-precontinuous, $S_{\gamma\vee\gamma'}$-continuous and $\beta_{\gamma\vee\gamma'}$-continuous).

Example 4.6. Let $X = \{a,b,c\}$ and $\tau = \{\varnothing, X, \{a\}, \{a,b\}\}$ and $\sigma = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}\}$. We define the operators $\gamma \vee \gamma'\colon \tau \to \mathcal{P}(X)$ by $\gamma(A) = \mathrm{Cl}(A)$ and $\gamma'(A) = \mathrm{Int}(\mathrm{Cl}(A))$ for all $A \in \tau$. Then $\tau_{\gamma\vee\gamma'} = \{\varnothing, X\}$. Define a function $f\colon (X,\tau) \to (Y,\sigma)$ as $f(a) = f(c) = a$, $f(b) = b$. Then f is both $S_{\gamma\vee\gamma'}$-continuous and $\gamma \vee \gamma'$-precontinuous, but it is neither $\gamma \vee \gamma'$-semicontinuous nor $B_{\gamma\vee\gamma'}$-continuous.

Example 4.7. Let $X = \{a, b, c, d\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ and $\sigma = \{\varnothing, Y, \{a\}, \{b\}, \{a, b\}\}$. We define the operations $\gamma, \gamma' \colon \tau \to \mathcal{P}(X)$ by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} \mathrm{Int}(\mathrm{Cl}(A)) & \text{if } A = \{a\}, \\ \mathrm{Cl}(A) & \text{if } A \neq \{a\}. \end{cases}$$

Then $\tau_{\gamma \vee \gamma'} = \{\varnothing, \{a\}, \{c\}, \{a, c\}, \{a, b, d\}, X\}$. Define a function $f \colon (X, \tau) \to (Y, \sigma)$ as $f(a) = f(c) = a$, $f(b) = f(d) = b$. Then f is $\beta_{\gamma \vee \gamma'}$-continuous, but it is not $\gamma \vee \gamma'$-semiprecontinuous.

Example 4.8. Let $X = \{a, b, c\}$ and $\tau = \{\varnothing, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ and $\sigma = \{\varnothing, X, \{a\}\}$. We define the operations $\gamma, \gamma' \colon \tau \to \mathcal{P}(X)$ by

$$A^{\gamma} = A^{\gamma'} = \begin{cases} \mathrm{Int}(\mathrm{Cl}(A)) & \text{if } A = \{a\}, \\ X & \text{if } A \neq \{a\}. \end{cases}$$

Then $\tau_{\gamma \vee \gamma'} = \{\varnothing, \{a\}, X\}$. Define a function $f \colon (X, \tau) \to (Y, \sigma)$ as $f(a) = f(c) = a$, $f(b) = b$. Then f is $\gamma \vee \gamma'$-α-continuous but it is not $C_{\gamma \vee \gamma'}$-continuous.

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