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New types of locally connected spaces via clopen set

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Abstract:

In this paper, we define and study a new type of connected spaces called λ_{co}-connected space. It is remarkable that the class of λ-connected spaces is a subclass of the class of λ_{co}-connected spaces. We discuss some characterizations and properties of λ_{co}-connected spaces, λ_{co} components and λ_{co}-locally connected spaces.

Keywords: λ_{co}-connected spaces; λ_{co}-components; λ_{co}-locally connected spaces.

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1. Introduction

Following [3] N. Levine, 1963, defined semi open sets. Similarly, S. F. Namiq [4], defined an operation λ on the family of semi open sets in a topological space called semi operation, denoted by s-operation; via this operation, in his study [7], he defined λ_{sc}-open set by using λ-open and semi closed sets, and also following [5], he defined λ_{co}-open set and investigated several properties of λ_{co}-derived, λ_{co}-interior and λ_{co}-closure points in topological spaces.

In the present article, we define the λ_{co}-connected space, discuss some characterizations and properties of λ_{co}-connected spaces, λ_{co}-components and λ_{co}-locally connected spaces and finally its relations with others connected spaces.

2. Preliminaries

In the entire parts of the present paper, a topological space is referred to by (X, τ) or simply by X. First, some definitions are recalled and results are used in this paper. For any subset A of X, the closure and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. Following [8], the researchers state that a subset A of X is regular closed if $A = \mathrm{Cl}(\mathrm{Int}(A))$. Similarly, following [3], a subset A of a space X is semi open if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a semi open set is called semi closed. The family of all semi open (resp. semi closed) sets in a space X is denoted by $\mathrm{SO}(X, \tau)$ or $\mathrm{SO}(X)$ (resp. $\mathrm{SC}(X, \tau)$ or $\mathrm{SC}(X)$. According to [1], a space X is stated to be s-connected, if it is not the union of two nonempty disjoint semi open subsets of X. We consider $\lambda{:}\mathrm{SO}(X) \to P(X)$ as a function defined on $\mathrm{SO}(X)$ into the power set of X, $P(X)$ and λ is called a semi-operation denoted by s-operation, if $V \subseteq \lambda(V)$, for each semi open set V. It is assumed that $\lambda(\emptyset) = \emptyset$ and $\lambda(X) = X$, for any s-operation. Let X be a space and λ $:\mathrm{SO}(X) \to P(X)$ be an s-operation, following [4], a subset A of X is called a λ-open set, which is equivalent to λ_s-open set [2], if for each $x \in A$, there exists a semi open set U such that $x \in U$ and $\lambda(U) \subseteq A$. The complement of a λ-open set is called a λ-closed. The family of all λ-open (resp., λ-closed) subsets of a space X is denoted by $\mathrm{SO}_\lambda(X, \tau)$ or $\mathrm{SO}_\lambda(X)$ (resp, $\mathrm{SC}_\lambda(X, \tau)$ or $\mathrm{SC}_\lambda(X)$). Following [4], a λ-open subset A of X is named a λ_c-open set, if for each $x \in A$, there exists a closed set F such that $x \in F \subseteq A$. The family of all λ_c-open (resp., λ_c-closed) subsets of a space X is denoted by $\mathrm{SO}_{\lambda_c}(X, \tau)$ or $\mathrm{SO}_{\lambda_c}(X)$ (resp, $\mathrm{SC}_{\lambda_c}(X, \tau)$ or $\mathrm{SC}_{\lambda_c}(X)$). Thus, a number of definitions are presented and some known results are reiterated which will be used in the sequel.

Definition 2.1. *[4] Let X be a space and $\lambda{:}SO(X) \to P(X)$ be an s-operation, then a subset A of X is called a λ-open set if for each $x \in A$ there exists a semi open set U such that $x \in U$ and $\lambda(U) \subseteq A$. The complement of a λ-open set is called λ-closed. The family of all λ-open (resp., λ-closed) subsets of a topological space (X, τ) is denoted by $SO_\lambda(X, \tau)$ or $SO_\lambda(X)$ (resp., $SC_\lambda(X, \tau)$ or $SC_\lambda(X)$).*

Definition 2.2. *[5] A λ-open subset A of X is called a λ_{co}-open (resp., λ_c-open [4]) set if for each $x \in A$, there exists a clopen (resp., closed) set F such that $x \in F \subseteq A$. The family of all λ_c-open (resp., λ_c-closed) subsets of a space X is denoted by $SO_{\lambda_c}(X, \tau)$ or $SO_{\lambda_c}(X)$ (resp $SC_{\lambda_c}(X, \tau)$ or $SC_{\lambda_c}(X)$). The family of all λ_{co}-open (resp., λ_{co}-closed) subsets of a space X is denoted by $SO_{\lambda_{co}}(X, \tau)$ or $SO_{\lambda_{co}}(X)$ (resp $SC_{\lambda_{co}}(X, \tau)$ or $SC_{\lambda_{co}}(X)$).*

Proposition 2.3. *[4],[5] For a space X, $SO_{\lambda_{co}}(X) \subseteq SO_{\lambda_c}(X) \subseteq SO_\lambda(X) \subseteq SO(X)$.*

Definition 2.4. *[2] Let X be a space, an s-operation λ is said to be s-regular if for every semi open sets U and V containing $x \in X$, there exists a semi open set W containing x such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V)$.*

Definition 2.5. *[6] A space X is said to be λ-connected if there does not exist a pair A, B of nonempty disjoint λ-open subset of X such that $X = A \cup B$, otherwise X is called λ-disconnected. In this case, the pair (A, B) is called a λ-disconnection of X.*

Following [5], we used some results:

Definition 2.6. *Let X be a space and A a subset of X. Then:*

1. *The λ_{co}-closure of A, denoted by $\lambda_{co}Cl(A)$ is the intersection of all λ_{co}-closed sets containing A.*
2. *The λ_{co}-interior of A, denoted by $\lambda_{co}Int(A)$ is the union of all λ_{co}-open sets of X contained in A.*
3. *A point $x \in X$ is said to be a λ_{co}-limit point of A if every λ_{co}-open set containing x contains a point of A different from x, and the set of all λ_{co}-limit points of A is called the λ_{co}-derived set of A, denoted by $\lambda_{co}D(A)$.*

Proposition 2.7. *For each point $x \in X$, $x \in \lambda_{co}Cl(A)$ if and only $V \cap A \neq \emptyset$, for every $V \in SO_{\lambda_{co}}(X)$ such that $x \in V$.*

Proposition 2.8. *Let $\{A_\alpha\}_{\alpha \in I}$ be any collection of λ_{co}-open sets in a topological space (X, τ), then $\cup_{\alpha \in I} A_\alpha$ is a λ_{co}-open set.*

Example 2.9. *Let $X = \{a, b, c\}$ and $\tau = P(X)$. We define an s-operation λ:$SO(X) \to P(X)$ as:*

$$\lambda(A) = \begin{cases} A & \text{if } A \neq \{a\}, \{b\}, \\ X & \text{otherwise}. \end{cases}$$

Now, we have $\{a, b\}$ and $\{b, c\}$ are λ_{co}-open sets, but $\{a, b\} \cap \{b, c\} = \{b\}$ is not λ_{co}-open.

Proposition 2.10. *Let λ be an s-operation and s-regular. If A and B are λ_{co}-open sets in X, then $A \cap B$ is also a λ_{co}-open set.*

Proposition 2.11. *Let X be a space and $A \subseteq X$. Then A is a λ_{co}-closed subset of X if and only if $\lambda_{co}D(A) \subseteq A$.*

Proposition 2.12. *For subsets A, B of a space X, the following statements are true.*

1. $A \subseteq \lambda_{co}Cl(A)$.
2. $\lambda_{co}Cl(A)$ *is a* λ_{co}*-closed set in* X.
3. $\lambda_{co}Cl(A)$ *is a smallest* λ_{co}*-closed set, containing* A.
4. A *is a* λ_{co}*-closed set if and only if* $A = \lambda_{co}Cl(A)$.
5. $\lambda_{co}Cl(\emptyset) = \emptyset$ and $\lambda_{co}Cl(X) = X$.
6. *If A and B are subsets of the space X with $A \subseteq B$. Then $\lambda_{co}Cl(A) \subseteq \lambda_{co}Cl(B)$.*
7. *For any subsets A, B of a space X. $\lambda_{co}Cl(A) \cup \lambda_{co}Cl(B) \subseteq \lambda_{co}Cl(A \cup B)$.*
8. *For any subsets A, B of a space X. $\lambda_{co}Cl(A \cap B) \subseteq \lambda_{co}Cl(A) \cap \lambda_{co}Cl(B)$.*

Proposition 2.13. *Let X be a space and $A \subseteq X$. Then $\lambda_{co}Cl(A) = A \cup \lambda_{co}D(A)$.*

3. λ_{co}-Connected Spaces

In this section, we define, study and characterize the λ_{co}-connected space, finally some of its properties are established.
We start this section with the following definitions.

Definition 3.1. *Let X be a space and $Y \subseteq X$. Then the class of λ_{co}-open sets in Y denoted by $SO_{\lambda_{co}}(Y)$, is defined in a natural way as: $SO_{\lambda_{co}}(Y) = \{Y \cap V : V \in SO_{\lambda_{co}}(X)\}$. That is, W is λ_{co}-open in Y if and only if $W = Y \cap V$, where V is a λ_{co}-open set in X. Thus, Y is a subspace of X with respect to λ_{co}-open set.*

Definition 3.2. *A space X is said to be λ_{co}-connected if there does not exist a pair A, B of nonempty disjoint λ_{co}-open subset of X such that $X = A \cup B$, otherwise X is called λ_{co}-disconnected. In this case, the pair (A, B) is called a λ_{co}-disconnection of X.*

Definition 3.3. *Let X be a space and λ:$SO(X) \to P(X)$ an s-operation, then the family $SO_{\lambda_{co}}(X)$ is called λ_{co}-indiscrete space if $SO_{\lambda_{co}}(X) = \{\emptyset, X\}$.*

Definition 3.4. *Let X be a space and λ:$SO(X) \to P(X)$ an s-operation then the family $SO_{\lambda_{co}}(X)$ is called a λ_{co}-discrete space if $SO_{\lambda_{co}}(X) = P(X)$.*

Example 3.5. *Every λ_{co}-indiscrete space is λ_{co}-connected.*

We give in below a characterization of λ_{co}-connected spaces, the proof of which is straight forward.

Theorem 3.6. *A space X is λ_{co}-disconnected (resp. λ_{co}-connected) if and only if there exists (resp., does not exist) a nonempty proper subset A of X, which is both λ_{co}-open and λ_{co}-closed in X.*

Theorem 3.7. *Every λ-connected space is λ_{co}-connected.*

Let X be λ-connected, then there does not exist a pair A, B of nonempty disjoint λ-open subset of X such that $X = A \cup B$, but every λ_{co}-open set is a λ-open set by Proposition 2.3, so there does not exist a pair A, B of nonempty disjoint λ_{co}-open subset of X such that $X = A \cup B$. Thus X is λ_{co}-connected.

The converse of Theorem 3.7, is not true in general as it is shown by the following example.

Example 3.8. *Let $X = \{a, b, c\}$, and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. We define an s-operation $\lambda : SO(X) \to P(X)$ as follows:*

$$\lambda(A) = \begin{cases} A & \text{if } A = \{a\}, \\ X & \text{otherwise} . \end{cases}$$

$SO(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$.
$SO_\lambda(X) = \{\emptyset, \{a\}, X\}$.
$SO_{\lambda_{co}}(X) = \{\emptyset, X\}$.
We have X is λ_{co}-connected, but it is not λ-connected.

Definition 3.9. *Let X be a space and $A \subseteq X$. The λ_{co}-boundary of A, denoted by $\lambda_{co}Bd(A)$, is defined as the set $\lambda_{co}Bd(A) = \lambda_{co}Cl(A) \cap \lambda_{co}Cl(X \setminus A)$.*

Theorem 3.10. *A space X is λ_{co}-connected if and only if every nonempty proper subspace has a nonempty λ_{co}-boundary.*

Suppose that a nonempty proper subspace A of a λ_{co}-connected space X has empty λ_{co}-boundary. Then A is λ_{co}-open and $\lambda_{co}\mathrm{Cl}(A) \cap \lambda_{co}\mathrm{Cl}(X \setminus A) = \emptyset$. Let p be a λ_{co}-limit point of A. Then $p \in \lambda_{co}\mathrm{Cl}(A)$, but $p \notin \lambda_{co}\mathrm{Cl}(X \setminus A)$. In particular $p \notin (X \setminus A)$ and so $p \in A$. Thus A is λ_{co}-closed and λ_{co}-open. By Theorem 3.6, X is λ_{co}-disconnected. This contradiction gives that A has a nonempty λ_{co}-boundary.
Conversely, suppose X is λ_{co}-disconnected. Then by Theorem 3.6, X has a proper subspace A which is both λ_{co}-closed and λ_{co}-open. Then $\lambda_{co}\mathrm{Cl}(A) = A$, $\lambda_{co}\mathrm{Cl}(X \setminus A) = (X \setminus A)$ and $\lambda_{co}\mathrm{Cl}(A) \cap \lambda_{co}\mathrm{Cl}(X \setminus A) = \emptyset$. So A has empty λ_{co}-boundary, a contradiction. Hence X is λ_{co}-connected. This completes the proof.

Theorem 3.11. *Let (A, B) be a λ_{co}-disconnection of a space X and C be a λ_{co}-connected subspace of X. Then C is contained in A or in B.*

Suppose that C is neither contained in A nor in B. Then $C \cap A$, $C \cap B$ are both nonempty λ_{co}-open subsets of C such that $(C \cap A) \cap (C \cap B) = \emptyset$ and $(C \cap A) \cup (C \cap B) = C$. This gives that $(C \cap A, C \cap B)$ is a λ_{co}-disconnection of C. This contradiction proves the theorem.

Theorem 3.12. *Let $X = \cup_{\alpha \in I} X_\alpha$, where each X_α is λ_{co}-connected and $\bigcap_{\alpha \in I} X_\alpha \neq \emptyset$. Then X is λ_{co}-connected.*

Suppose on the contrary that (A, B) is a λ_{co}-disconnection of X. Since each X_α is λ_{co}-connected, therefore by Theorem 3.11, $X_\alpha \subseteq A$ or $X_\alpha \subseteq A$. Since $\bigcap_{\alpha\in I} X_\alpha \neq \emptyset$, therefore all X_α are contained in A or in B. This gives that, if $X \subseteq A$, then $B = \emptyset$ or if $X \subseteq B$, then $A = \emptyset$. This contradiction proves that X is λ_{co}-connected. Which completes the proof.

Using Theorem 3.12, we give a characterization of λ_{co}-connectedness as follows:

Theorem 3.13. *A space X is λ_{co}-connected if and only if for every pair of points x, y in X, there is a λ_{co}-connected subset of X, which contains both x and y.*

The necessity is immediate since the λ_{co}-connected space itself contains these two points. For the sufficiency, suppose that for any two points x, y; there is a λ_{co}-connected subspace $C_{(x,y)}$ of X such that $x, y \in C_{(x,y)}$. Let $a \in X$ be a fixed point and $\{C_{(a,x)} : x \in X\}$ a class of all λ_{co}-connected subsets of X, which contain the points a, x. Then $X = \cup_{x\in X} C_{(a,x)}$ and $\cap_{x\in X} C_{(a,x)} \neq \emptyset$. Therefore, by Theorem 3.12, X is λ_{co}-connected. This completes the proof.

Theorem 3.14. *Let C be a λ_{co}-connected subset of a space X and $A \subseteq X$ such that $C \subseteq A \subseteq \lambda_{co}Cl(C)$. Then A is λ_{co}-connected.*

It is sufficient to show that $\lambda_{co}\mathrm{Cl}(C)$ is λ_{co}-connected. On the contrary, suppose that $\lambda_{co}\mathrm{Cl}(C)$ is λ_{co}-disconnected. Then there exists a λ_{co}-disconnection (H, K) of $\lambda_{co}\mathrm{Cl}(C)$. That is, $H \cap C$, $K \cap C$ are λ_{co}-open sets in C such that $(H \cap C) \cap (K \cap C) = (H \cap K) \cap C = \emptyset$ and $(H \cap C) \cup (K \cap C) = (H \cup K) \cap C = C$. This gives that $(H \cap C, K \cap C)$ is a λ_{co}-disconnection of C, a contradiction. This proves that $\lambda_{co}\mathrm{Cl}(C)$ is λ_{co}-connected.

4. λ_{co}-components and λ_{co}-locally connected spaces

In this section a new types of λ_{co}-component of a space X and λ_{co}-locally connected space are defined, studied and characterized and finally some of its properties are established.

Definition 4.1. *A maximal λ_{co}-connected subset of a space X is called a λ_{co}-component of X. If X itself is λ_{co}-connected, then X is the only λ_{co}-component of X.*

Next we study the properties of λ_{co}-components of a space X.

Theorem 4.2. *Let (X, τ) be a topological space. Then:*

1. *For each $x \in X$, there is exactly one λ_{co}-component of X containing x.*
2. *Each λ_{co}-connected subset of X is contained in exactly one λ_{co}-component of X.*
3. *A λ_{co}-connected subset of X, which is both λ_{co}-open and λ_{co}-closed is a λ_{co}-component, if λ is s-regular.*
4. *Every λ_{co}-component of X is λ_{co}-closed in X.*

(1)-Let $x \in X$ and $\{C_\alpha : \alpha \in I\}$ be a class of all λ_{co}-connected subsets of X containing x. Put $C = \bigcup_{\alpha \in I} C_\alpha$, then by Theorem 3.12, C is λ_{co}-connected and $x \in X$. Suppose $C \subseteq C^1$, for some λ_{co}-connected subset C^1 of X. Then $x \in C^1$ and hence C^1 is one of the C_α's and hence $C^1 \subseteq C$. Consequently $C = C^1$. This proves that C is a λ_{co}-component of X, which contains x.
(2)-Let A be a λ_{co}-connected subset of X, which is not a λ_{co}-component of X. Suppose that C_1, C_2 are λ_{co}-components of X such that $A \subseteq C_1$, $A \subseteq C_2$. Since $C_1 \cap C_2 \neq \emptyset$, $C_1 \cup C_2$ is another λ_{co}-connected set which contains C_1 as well as C_2, this contradicts the fact that C_1 and C_2 are λ_{co}-components. This proves that A is contained in exactly one λ_{co}-component of X.
(3)-Suppose that A is a λ_{co}-connected subset of X which is both λ_{co}-open and λ_{co}-closed. By (2), A is contained in exactly one λ_{co}-component C of X. If A is a proper subset of C, and since λ is s-regular, therefore $C = (C \cap A) \cup (C \cap (X \setminus A))$ is a λ_{co}-disconnection of C, a contradiction. Thus, $A = C$.
(4)-Suppose a λ_{co}-component C of X is not λ_{co}-closed. Then, by Theorem 3.14, $\lambda_{co}\mathrm{Cl}(A)$ is λ_{co}-connected containing a λ_{co}-component C of X. This implies $C = \lambda_{co}\mathrm{Cl}(A)$ and hence C is λ_{co}-closed. This completes the proof.

We introduce the following definition

Definition 4.3. *A space X is said to be locally λ_{co}-connected if for any point $x \in X$ and any λ_{co}-open set U containing x, there is a λ_{co}-connected and λ_{co}-open set V such that $x \in V \subseteq U$.*

Theorem 4.4. *A λ_{co}-open subset of λ_{co}-locally connected space is λ_{co}-locally connected.*

Let U be a λ_{co}-open subset of a λ_{co}-locally connected space X. Let $x \in U$ and V a λ_{co}-open nbd of x in U. Then V is a λ_{co}-open neighborhood of x in X. Since X is λ_{co}-locally connected, therefore there exists a λ_{co}-connected, λ_{co}-open neighborhood W of x such that $x \in W \subseteq V$. So that W is also a λ_{co}-connected and λ_{co}-open neighborhood x in U such that $x \in W \subseteq U \subseteq V$ or $x \in W \subseteq V$. This proves that U is λ_{co}-locally connected.

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