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Existence of (N, λ)-Periodic Solutions for Abstract Fractional Difference Equations

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Abstract. We establish sufficient conditions for the existence and uniqueness of (N, λ)-periodic solutions for the following abstract model:

$$\Delta^{\alpha} u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{Z},$$

where $0 < \alpha \leq 1$, A is a closed linear operator defined in a Banach space X, Δ^{α} denotes the fractional difference operator in the Weyl-like sense, and f satisfies appropriate conditions.

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1. Introduction

In this article, we investigate the existence of a class of solutions for the abstract fractional difference equation

$$\Delta^{\alpha} u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{1.1}$$

called (N, λ)-periodic solutions. In (1.1), A is a possibly unbounded operator defined on a Banach space X and $f : \mathbb{Z} \times X \to X$ is given. This class of (N, λ)-periodic functions was introduced in the reference [6] as the discrete counterpart of the notion of (ω, c)-periodic functions [10], a notion that has been studied by various authors, see, e.g., [8, 9, 15–19, 26] and [30]. It is worth noting that class of (N, λ)-periodic functions contains the classes of discrete periodic ($\lambda = 1$), discrete anti-periodic ($\lambda = -1$), discrete Bloch-periodic ($\lambda = e^{ikN}, k \in \mathbb{Z}$ fixed), and unbounded functions.

Existence and uniqueness of (N, λ)-periodic solutions for *scalar models*, such as Volterra difference equations with infinite delay, were recently investigated in [6]. Anticipating a growing theoretical and practical interest in this class of solutions, we study in this article the existence and uniqueness

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of solutions for the abstract Cauchy problem (1.1). This problem extends the class of models under study so far, to the broader context of partial difference-differential equations, that is *vector-valued models,* since A could be an unbounded operator defined in a Banach space X, e.g., the Laplacian operator on $L^2(\mathbb{R}^d)$. In fact, as we will reveal in this investigation, when A is the generator of a C_0-semigroup that is strictly contractive, plus other conditions on the non-linear term, then the existence of (N, λ)-periodic solutions for (1.1) can be guaranteed.

Discrete fractional calculus has received considerable interest in recent years due to its interaction with applied mathematics and computation. We refer to the articles [14] for applications to interval-valued systems, [4] for applications to chaotic systems with short memory and image encryption, [13] for applications to variable-order fractional discrete-time recurrent neural networks, [31] for applications to image enhancement, and [32] for applications to Lyapunov functions for Riemann-Liouville-like fractional difference equations.

The existence of solutions for the abstract model (1.1) began to be studied in the articles [21] and [12] in its linearized form. Subsequently, maximal regularity in Lebesgue spaces of sequences was studied in [22]. In case A is bounded, weighted bounded solutions were studied in [24]. In [2], the existence of almost automorphic mild solutions was studied.

However, the existence and uniqueness of (N, λ)-periodic solutions is an open topic that deserves to be investigated. The objective of this work is to solve this problem.

As methods, we use the technique of *resolvent sequences of operators,* a tool that was introduced in 2017 by Lizama [21] and has been used in several articles since then. Using this method, an explicit representation of the solution for (1.1) can be obtained, which allows the use of several fixed point theorems.

Recently, it has been shown that resolvent sequences of operators can be related to each other by means of a subordination principle [7], at least when $0 < \alpha \leq 1$. In this work, we will refine the results in [7] observing that the only necessary condition to obtain the existence of resolvent sequences of operators is: $1 \in \rho(A)$, the resolvent set of A. This crucial observation is stated in Theorem 3.1 below, and follows from the subordination principle which we will generalize here using an extension of the discrete version of the Lévy α-stable distribution.

Therefore, our main result regarding the solubility of (1.1) can be proved and it says the following: suppose that $1 \in \rho(A)$ and

$$r_A := \|(I - A)^{-1}\| < 1. \tag{1.2}$$

Assume that there exist $(N, \lambda) \in \mathbb{N} \times (\mathbb{C} \backslash \mathbb{D})$ and a constant $L > 0$, such that $f(n + N, \lambda x) = \lambda f(n, x)$ for all $(n, x) \in \mathbb{Z} \times X$ and

$$\|f(n, x) - f(n, y)\| \; \leq L\|x - y\|,$$

for all $x, y \in X$ and all $n \in \mathbb{Z}$. If

$$L < \left(1 - \frac{1}{|\lambda|^{1/N}}\right)^{\alpha} + \left(\frac{1}{r_A} - 1\right), \tag{1.3}$$

then Eq. (1.1) has a unique (N, λ)-periodic solution in a mild sense.

This article is organized as follows: Section 2 is devoted to the preliminaries about the notion of fractional order difference operator Δ^{α} that we will use. We will also remember the notion of Mittag–Leffler sequence and of (μ, ν)-resolvent sequence of operators. We end this section by remembering the definition of (N, λ)-periodic sequence. Section 3 introduces the notion of a scaled Wright sequence and lists some of its main properties. Also in this section, we establish the important Theorem 3.1, showing that the condition $1 \in \rho(A)$ is sufficient for the existence of (μ, ν)-resolvent sequence of operators. Then, we prove the Theorem 3.4 which says that under the condition (1.2), the summability of (ν, ν)-resolvent sequences can be ensured. Using this relevant fact, we solve in Section 4 the problem of existence and uniqueness of (N, λ)-periodic solutions for the equation (1.1). See Theorem 4.5. Finally, an example is given where A is the one-dimensional Laplacian in $X = L^2(0, 1)$.

2. Preliminaries

Let X be a complex Banach space with norm $\|\cdot\|$ and $\mathcal{B}(X)$ denotes the Banach space of all bounded operators defined on X. For a real number a, we denote $\mathbb{N}_a := \{a, a+1, a+2, \dots\}$, and when $a = 1$, we write $\mathbb{N}$. We recall that the finite discrete convolution $*$ of two sequences $f, g : \mathbb{N}_0 \to X$ is defined by

$$(f * g)(n) := \sum_{j=0}^{n} f(n-j) g(j), \quad n \in \mathbb{N}_0.$$

We denote by $s(\mathbb{Z}, X)$ the vector space consisting of all vector-valued sequences $f : \mathbb{Z} \to X$. For $f \in s(\mathbb{Z}, X)$, we recall that the forward difference operator $\Delta : s(\mathbb{Z}, X) \to s(\mathbb{Z}, X)$ is defined by

$$\Delta f(n) := f(n+1) - f(n), \quad n \in \mathbb{Z}.$$

On the other hand, for an arbitrary $\alpha \in \mathbb{C} \backslash \{0, -1, -2, ...\}$ *the Cesàro sequence* $\{k^{\alpha}(n)\}_{n \in \mathbb{N}_0}$, introduced in [21] (see also [33]), is defined by

$$k^{\alpha}(n) := \frac{\Gamma(\alpha + n)}{\Gamma(\alpha) n!}, \quad n \in \mathbb{N}_0. \tag{2.1}$$

In case $\alpha = 0$, we define $k^0(n) := \delta_0(n)$, the Kronecker delta.

The following equality and estimate holds: for $\alpha > 0$

$$k^{\alpha}(n) = \frac{1}{n^{1-\alpha}\Gamma(\alpha)} \left(1 + O\left(\frac{1}{n}\right)\right)$$

and for $0 < \alpha < 1$,

$$\frac{1}{\Gamma(\alpha)(n+1)^{1-\alpha}} < k^{\alpha}(n) < \frac{1}{\Gamma(\alpha) n^{1-\alpha}}, \qquad n \in \mathbb{N}. \tag{2.2}$$

Furthermore, given $\alpha, \beta > 0$, the sequence k^α satisfies the semigroup property in $\mathbb{N}_0$, that is

$$(k^\alpha * k^\beta)(n) = k^{\alpha+\beta}(n), \quad \forall\, n \in \mathbb{N}_0; \tag{2.3}$$

see [3, Sect. 2]. Given $\alpha > 0$, we define the set

$$\ell^1_\alpha(\mathbb{Z}, X) := \left\{ f \in s(\mathbb{Z}, X) \, : \, \sum_{n=-\infty}^{\infty} \|n^{\alpha-1} f(n)\| < \infty \right\}.$$

It is clear that $\ell^1_\alpha(\mathbb{Z}, X)$ is a Banach space under the norm $\|f\|_{\ell^1_\alpha} := \sum_{n=-\infty}^{\infty} \|n^{\alpha-1} f(n)\|$.

If $\alpha = 1$, then we simply write $\ell^1(\mathbb{Z}, X)$. Now, suppose that $0 < \alpha \leq 1$. Observe that, if $f \in \ell^1(\mathbb{Z}, X)$, then

$$\|f\|_{\ell^1_\alpha} := \sum_{n=-\infty}^{\infty} \|n^{\alpha-1} f(n)\| < \sum_{n=-\infty}^{\infty} \|f(n)\| < \infty.$$

Hence, $\ell^1(\mathbb{Z}, X) \subset \ell^1_\alpha(\mathbb{Z}, X)$ for $0 < \alpha \leq 1$.

The theory and applications of operators defined by means of the Cesàro sequences defined on $\mathbb{N}_0$ have been worked in different investigations (see, for example, [3,7,11,20,21,27]). In this paper, we will work with the following fractional sum operator defined on $\mathbb{Z}$ in the reference [2].

Definition 2.1. [2] Given $0 < \alpha < 1$ the α-th fractional sum operator $\Delta^{-\alpha} : \ell^1(\mathbb{Z}, X) \to s(\mathbb{Z}, X)$ is defined by means of the formula

$$\Delta^{-\alpha} f(n) := \sum_{j=-\infty}^{n} k^\alpha(n-j) f(j), \quad f \in \ell^1(\mathbb{Z}, X). \tag{2.4}$$

Remark 2.2. Note that, for $\alpha = 0$, $\Delta^{-\alpha} f(n) = f(n)$.

The next definition about fractional differences operators in the sense of Riemman–Liouville and Caputo was introduced by Abadias and Lizama in [2].

Definition 2.3. Let $0 < \alpha < 1$ and $f \in \ell^1(\mathbb{Z}, X)$. The Caputo fractional difference operator of order α is defined by

$${}_c\Delta^\alpha f(n) := \Delta^{-(1-\alpha)} \Delta f(n),$$

and the Riemann–Liouville fractional difference operator of order α is defined by

$${}_R\Delta^\alpha f(n) := \Delta \Delta^{-(1-\alpha)} f(n). \tag{2.5}$$

Given $f \in \ell^1_\alpha(\mathbb{Z}, X)$, it was proved in [2] that

$${}_R\Delta^\alpha f(n) = {}_c\Delta^\alpha f(n), \;\; n \in \mathbb{Z}.$$

Therefore, from now on, we will simply denote by Δ^α either ${}_R\Delta^\alpha$ or ${}_c\Delta^\alpha$.

Now, we recall the notion of *Mittag–Leffler sequence* defined and studied in the references [7,21,23,27].

Let $\alpha, \beta > 0$ and $\sigma \in \mathbb{C}$ be such that $|\sigma| < 1$. We define

$$\mathcal{E}_{\alpha,\beta}(\sigma, n) := \sum_{j=0}^{\infty} \sigma^j k^{\alpha j+\beta}(n), \ \ n \in \mathbb{N}_0. \tag{2.6}$$

Note that the series on the right-hand side of (2.6) converges by (2.2). Furthermore, the Z-transform of the Mittag–Leffler sequence exists for $|z| > 1$ (see [7]), and is given by

$$\sum_{j=0}^{\infty} \mathcal{E}_{\alpha,\beta}(\sigma, j) z^{-j} = \left(\frac{z}{z-1}\right)^{\beta} \left(1 - \sigma \left(\frac{z}{z-1}\right)^{\alpha}\right)^{-1}. \tag{2.7}$$

Motivated by [7, (the proof of) Proposition 4.6], we get the following result regarding the asymptotic behavior of the Mittag–Leffler sequence.

Theorem 2.4. *Let $0 < \alpha < 2$, $\beta, C > 0$ and $\sigma \in \mathbb{C}$ be such that $|\sigma| < 1$. The Mittag–Leffler sequence (2.6) satisfies the following inequality:*

$$|\mathcal{E}_{\alpha,\beta}(-\sigma, n)| \leq \frac{C}{1 + \sigma n^{\alpha}}, \qquad n \in \mathbb{N}_0.$$

Proof. By (2.6), the inequality (2.2), and [28, Theorem 1.6], the result follows. □

Next, we recall the concept of discrete (α, ϱ)-resolvent sequence defined in [7, Sect. 4, Definition 4.4]. Also, useful results related with this definition are given.

Definition 2.5. Let $\varrho, \alpha > 0$ be given and A be a closed linear operator with domain $D(A)$ defined on a Banach space X. An operator-valued sequence $\{S_{\alpha,\varrho}(n)\}_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ is called a discrete (α, ϱ)-resolvent sequence generated by A if it satisfies the following conditions:

(i) $S_{\alpha,\varrho}(n)x \in D(A)$ for all $x \in X$ and $S_{\alpha,\varrho}(n)Ax = AS_{\alpha,\varrho}(n)x$ for each $n \in \mathbb{N}_0$ and $x \in D(A)$;

(ii) $S_{\alpha,\varrho}(n)x = k^{\varrho}(n)x + A(k^{\alpha} * S_{\alpha,\varrho})(n)x$ for all $n \in \mathbb{N}_0$ and each $x \in X$.

We finish this section recalling the notion of (N, λ)-periodic sequences and their main properties. The notion of (N, λ)-periodic sequences was introduced in [6] as a discrete counterpart of the concept of (ω, c)-periodic functions defined in [10].

Definition 2.6 [6]. A vector-valued function $f : \mathbb{Z} \to X$ is called (N, λ)-periodic discrete function (or (N, λ)-periodic sequence) if there exist $N \in \mathbb{N}$ and $\lambda \in \mathbb{C}\setminus\{0\}$, such that $f(n + N) = \lambda f(n)$ for all $n \in \mathbb{Z}$. N is called the λ-period of f. The collection of those sequences with the same λ-period N will be denoted by $\mathbb{P}_{N\lambda}(\mathbb{Z}, X)$.

The following result is central for the theory.

Proposition 2.7 [6]. *A function f is (N, λ)-periodic discrete function if and only if there exists $u \in \mathbb{P}_N(\mathbb{Z}, X)$, such that*

$$f(n) = \lambda^{\wedge}(n)u(n), \qquad \textit{for all} \ \ n \in \mathbb{Z}, \tag{2.8}$$

where $\lambda^{\wedge}(n) := \lambda^{n/N}$.

The vector-valued space of sequences $\mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ is a Banach space with the norm

$$\|f\|_{N\lambda} := \max_{n\in[0,N]} \|\lambda^\wedge(-n)f(n)\|. \tag{2.9}$$

3. The Discrete Scaled Wright Function and Summability of Resolvent Sequences

In [7], the authors introduced a discrete version of the Lévy α-stable distribution which can be defined as

$$l_\alpha(n,j) = \sum_{i=0}^{j} \binom{j}{i} (-1)^i k^{-\alpha i}(n), \quad 0<\alpha<1, \quad n,j \in \mathbb{N}_0. \tag{3.1}$$

The sequence l_α is a probability density function in n, which means that

$$0 \le l_\alpha(n,j) \qquad \text{and} \qquad \sum_{n=0}^{\infty} l_\alpha(n,j) = 1. \tag{3.2}$$

This representation of the discrete Lévy function allowed to establish a subordination principle which relates a discrete (α,ϱ)-resolvent sequence with a C-semigroup generated by a given closed linear operator A defined on a Banach space X (see [7]). The following result is a consequence of this fact.

Theorem 3.1. *Let $0<\alpha\le\varrho\le 1$ be given. Let A be a closed and linear operator defined on a Banach space X, such that $1\in\rho(A)$. Then, the family*

$$S_{\alpha,\varrho}(n)x = \sum_{j=0}^{\infty} (k^{\varrho-\alpha} * l_\alpha(\cdot,j))(n)(I-A)^{-(j+1)}x, \quad n\in\mathbb{N}_0,\ x\in X \tag{3.3}$$

is a discrete (α,ϱ)-resolvent sequence generated by A.

Proof. By hypothesis, $C:=(I-A)^{-1}$ exists and the operator $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ given by $\mathcal{T}(n) = (I-A)^{-(n+1)}$ is bounded on X. On the other hand

$$\mathcal{T}(n) = (I-A)^{-(n+1)} = C^{-(n-1)}\mathcal{T}(1)^n.$$

Hence, the operator $\{\mathcal{T}(n)\}_{n\in\mathbb{N}_0}$ is a the discrete C-semigroup (see [7]). Thus, the result follows from Theorem 4.5 of [7]. □

The concept of scaled Wright function in the continuous case was introduced by Abadias and Miana in [1]. Motivated by the above theorem, we propose in this paper the following definition.

Definition 3.2. Let $0<\alpha<1$ and $0\le\beta$ be given. For $n\in\mathbb{N}_0$, the discrete scaled Wright function $\varphi_{\alpha,\beta}$ is defined by

$$\varphi_{\alpha,\beta}(n,j) := \sum_{i=0}^{j} \binom{j}{i} (-1)^i k^{\beta-\alpha i}(n). \tag{3.4}$$

Some properties of the discrete scaled Wright function can be deduced from those properties of the Lévy α-stable distribution. They are stated in the following remark.

Remark 3.3. (i) Note that, by (2.3) and (3.1)

$$\varphi_{\alpha,\beta}(n,j) = (k^{\beta} * l_{\alpha}(\cdot,j))(n)$$

and, in particular, $\varphi_{\alpha,0}(n,j) = l_{\alpha}(n,j)$.

(ii) Since the sequence l_{α} is non-negative, then the discrete scaled Wright function $\varphi_{\alpha,\beta}$ is non-negative.

(iii) The formula (3.3) can be written as

$$S_{\alpha,\varrho}(n) := \sum_{j=0}^{\infty} \varphi_{\alpha,\varrho-\alpha}(n,j)\mathcal{T}(j). \tag{3.5}$$

(iv) By Proposition 4.6 of [7], we have

$$\sum_{j=0}^{\infty} \varphi_{\alpha,\beta}(n,j)\,(1+\lambda)^{-(j+1)} = \mathcal{E}_{\alpha,\alpha+\beta}(-\lambda,n),\ n \in \mathbb{N},\ |\lambda| < 1.$$

(v) Let $A = \omega \in \mathbb{C}$ and $|w| < 1$. In this case, the discrete C-semigroup generated by A is given by $\mathcal{T}(n) = (1-\omega)^{-(n+1)}$. Then, by Theorem 3.1, A generates discrete (α,α)-resolvent and $(\alpha,1)$-resolvent sequences given by

$$S_{\alpha,\alpha}(n) = \textstyle\sum_{j=0}^{\infty} \varphi_{\alpha,0}(n,j)\mathcal{T}(j) = \mathcal{E}_{\alpha,\alpha}(\omega,n), \quad n \in \mathbb{N}_0, \tag{3.6}$$

and

$$S_{\alpha,1}(n) = \textstyle\sum_{j=0}^{\infty} \varphi_{\alpha,1-\alpha}(n,j)\mathcal{T}(j) = \mathcal{E}_{\alpha,1}(\omega,n), \quad n \in \mathbb{N}_0, \tag{3.7}$$

where we have used (iii) and (iv).

We recall that an operator-valued sequence $\{S(n)\}_{n\in\mathbb{N}_0} \in \mathcal{B}(X)$ is said to be *summable* if

$$\| S \|_1 := \sum_{n=0}^{\infty} \|S(n)\| < \infty.$$

We finish this section with a useful result which is a direct consequence of the above considerations.

Theorem 3.4. *Let A be a closed linear operator and suppose that $1 \in \rho(A)$ and*

$$\|(I-A)^{-1}\| < 1. \tag{3.8}$$

Then, A generates a summable discrete (α,α)-resolvent sequence $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$.

Proof. Since $1 \in \rho(A)$, then by Theorem 3.1, the family

$$S_{\alpha,\alpha}(n)x = \sum_{j=0}^{\infty} \varphi_{\alpha,0}(n,j)(I-A)^{-(j+1)}x, \quad n \in \mathbb{N}_0,\ x \in X$$

is a discrete (α,α)-resolvent sequence generated by A. We will prove that it is summable. Indeed, since $0 \le \varphi_{\alpha,0}(n,j) \le 1$ for $j \in \mathbb{N}_0$, then

$$\sum_{n=0}^{\infty} \|S_{\alpha,\alpha}(n)\| \le \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \varphi_{\alpha,0}(n,j)\|(I-A)^{-(j+1)}\| \le \sum_{j=0}^{\infty} \|(I-A)^{-(j+1)}\|$$
$$\le \|(I-A)^{-1}\| \sum_{j=0}^{\infty} \|(I-A)^{-1}\|^{j} < \infty.$$

□

The following example provide concrete conditions on A under which the condition (3.8) holds.

Example. Let A be the generator of a C_0-semigroup strictly contractive. For instance, on $X{:=}L^1(\mathbb{R})$, we define

$$(T(t)f)(s) = \begin{cases} \beta f(t+s) \; if \; s \in [-t,0], \\ \\ f(t+s) \qquad \text{otherwise}, \end{cases}$$

where $0 < \beta < 1$ is arbitrary. Then, $T(t)$ is a C_0-semigroup and $\|T(t)\| = \beta < 1$ (since $\|T(t)\mathbb{1}_{[0,t]}\| = \beta\|\mathbb{1}_{[0,t]}\|$).

We deduce that $1 \in \rho(A)$ and $\|(I-A)^{-1}\| < 1$. Indeed,

$$\|(I-A)^{-1}\| = \|\int_0^{\infty} e^{-t}T(t)dt\| \le \int_0^{\infty} e^{-t}\|T(t)\|dt < \beta < 1.$$

The last part of the earlier example shows the following result.

Corollary 3.5. *Let A be the generator of a C_0-semigroup strictly contractive, then $1 \in \rho(A)$ and $\|(I-A)^{-1}\| < 1$.*

4. (N,λ)-Periodic Solutions for Fractional Difference Equations on $\mathbb{Z}$

In this section, we study regularity of solutions to the linear fractional difference equation

$$\Delta^{\alpha} u(n) = Au(n+1) + g(n), \quad n \in \mathbb{Z},$$

and the non-linear fractional equation

$$\Delta^{\alpha} u(n) = Au(n+1) + f(n,u(n)), \quad n \in \mathbb{Z}.$$

in $\mathbb{P}_{N\lambda}(\mathbb{Z},X)$, where A be a closed linear operator with domain $D(A)$ defined on X.

4.1. The linear case

Let $0 < \alpha \leq 1$ and A be a closed linear operator with domain $D(A)$ defined on a Banach space X. We consider the linear fractional difference equation

$$\Delta^\alpha u(n) = Au(n+1) + g(n), \quad n \in \mathbb{Z}. \tag{4.1}$$

We recall from [2, Definition 4.1] that a sequence $u \in \ell^1(\mathbb{Z}, X)$ is called a *strong solution* for Eq. (4.1) if $u(n) \in D(A)$ for all $n \in \mathbb{Z}$ and u satisfies (4.1).

Definition 4.1 [2]. Let A be the generator of a discrete (α, α)-resolvent family $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ and $g : \mathbb{Z} \longrightarrow X$. The sequence

$$u(n) = \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j)g(j), \quad n \in \mathbb{Z} \tag{4.2}$$

is called a *mild solution* for Eq. (4.1) if $m \to S_{\alpha,\alpha}(m)g(n-m)$ is summable on $\mathbb{N}_0$ for each $n \in \mathbb{Z}$.

Note that if $g \in \ell^1(\mathbb{Z}, D(A))$, then each mild solution is a strong one; see [2, Theorem 4.2].

In the following theorem, we establish the existence of (N, λ)-periodic mild solutions for Eq. (4.1).

Theorem 4.2. *Let $0 < \alpha \leq 1$. Assume that A be a closed linear operator defined on a Banach space X, $1 \in \rho(A)$ and*

$$\|(I-A)^{-1}\| < 1.$$

If $g \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$, then there is an (N, λ)-periodic mild solution of (4.1) given by the sequence

$$u(n) := \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j)g(j), \quad n \in \mathbb{Z}, \tag{4.3}$$

where $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ is discrete (α, α)-resolvent sequence defined in (3.3).

Proof. By Theorem 3.4, A generates a summable discrete (α, α)-resolvent sequence $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}}$ given by

$$S_{\alpha,\alpha}(n)x = \sum_{j=0}^{\infty} \varphi_{\alpha,0}(n, j)(I-A)^{-(j+1)}x, \quad n \in \mathbb{N}_0, \ x \in X.$$

Since g is bounded and $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ is summable, it follows that the sequence u is a mild solution of (4.1).

It remains to prove that $u \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$. Indeed,

$$\begin{aligned} u(n+N) &= \sum_{j=-\infty}^{n+N-1} S_{\alpha,\alpha}(n+N-1-j)g(j) = \sum_{p=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-p)g(p+N) \\ &= \lambda \sum_{p=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-p)g(p) = \lambda u(n), \end{aligned}$$

getting that $u \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$.

$\square$

4.2. The Semilinear Case

In this subsection, we consider the following fractional difference equation:

$$\Delta^\alpha u(n) = Au(n+1) + f(n, u(n)), \quad n \in \mathbb{Z}, \tag{4.4}$$

where $0 < \alpha \leq 1$, A satisfies the hypotheses in Theorem 3.4 and f satisfies suitable conditions.

Inspired in the solution of the linear case, we give the following definition of mild solution for the semilinear case.

Definition 4.3. Let A be the generator of a discrete (α, α)-resolvent family $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ and $f : \mathbb{Z} \times X \longrightarrow X$. We say that a sequence $u : \mathbb{Z} \longrightarrow X$ is a (N, λ)-periodic mild solution of (4.4) if $u \in P_{N\lambda}(\mathbb{Z}, X)$ satisfies

$$u(n) = \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j) f(j, u(j)), \quad n \in \mathbb{Z}, \tag{4.5}$$

where $m \to S_{\alpha,\alpha}(m) f(n-m, x)$ is summable on $\mathbb{N}_0$ for each $n \in \mathbb{Z}$.

Let $f : \mathbb{Z} \times X \to X$, $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ and denote by $\mathcal{N}(\phi)(\cdot) := f(\cdot, \phi(\cdot))$ the Nemytskii discrete composition operator.

To prove the main theorem, we will need to recall the following.

Theorem 4.4 [6]. *Let $f : \mathbb{Z} \times X \to X$. Then, the following assertions are equivalent:*

(i) *For every $\phi \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$, we have that $\mathcal{N}(\phi)$ is (N, λ)-periodic discrete.*
(ii) *f is N-periodic in the first variable and homogeneous in the second variable, that is $f(n+N, \lambda x) = \lambda f(n, x)$ for all $(n, x) \in \mathbb{Z} \times X$.*

Let $\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. The following is our main result.

Theorem 4.5. *Let $f : \mathbb{Z} \times X \to X$ be given and let A be a closed linear operator defined on a Banach space X, such that $1 \in \rho(A)$ and*

$$r_A := \|(I - A)^{-1}\| < 1. \tag{4.6}$$

Assume the following conditions:

H$_1$. *There exists $(N, \lambda) \in \mathbb{N} \times (\mathbb{C} \backslash \mathbb{D})$, such that $f(n+N, \lambda x) = \lambda f(n, x)$ for all $(n, x) \in \mathbb{Z} \times X$.*

H$_2$. *There exists a constant $L > 0$, such that*

$$\|f(n, x) - f(n, y)\| \leq L\|x - y\|,$$

for all $x, y \in X$ and all $n \in \mathbb{Z}$.

H$_3$. *The constant L in H_2 is such that*

$$L < \left(1 - \frac{1}{|\lambda|^{1/N}}\right)^\alpha + \left(\frac{1}{r_A} - 1\right).$$

Then, Eq. (4.4) has a unique (N, λ)-periodic mild solution.

Proof. First, let us define the operator $G : \mathbb{P}_{N\lambda}(\mathbb{Z}, X) \to \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ by

$$G(u)(n) := \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j) f(j, u(j)).$$

Let $u \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$ and $g(n) := f(n, u(n))$. By H_1 and Theorem 4.4 we get that $g \in P_{N,\lambda}(\mathbb{Z}, X)$. As in the linear case, we can see that $G(u) \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$. It follows that G is well defined. Now, for $u, v \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$

$$\begin{aligned}&\|G(u) - G(v)\|_{N\lambda}\\&\quad = \max_{n\in[0,N]} \|\lambda^\wedge(-n-1) \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j)\,[f(j,u(j)) - f(j,v(j))]\,\|,\end{aligned}$$

where we have by H_2 that

$$\begin{aligned}&\|\lambda^\wedge(-n-1) \sum_{j=-\infty}^{n-1} S_{\alpha,\alpha}(n-1-j)\,[f(j,u(j)) - f(j,v(j))]\,\|\\
&= \|\sum_{j=-\infty}^{n-1} \lambda^\wedge(-(n-1-j)) S_{\alpha,\alpha}(n-1-j)\lambda^\wedge(-j)\,[f(j,u(j)) - f(j,v(j))]\,\|\\
&< \sum_{j=-\infty}^{n-1} |\lambda|^\wedge(-(n-1-j))|S_{\alpha,\alpha}(n-1-j)||\lambda|^\wedge(-j)\|\,[f(j,u(j)) - f(j,v(j))]\,\|\\
&< L \sum_{j=-\infty}^{n-1} |\lambda|^\wedge(-(n-1-j))|S_{\alpha,\alpha}(n-1-j)|\|\lambda^\wedge(-j)\,[u(j) - v(j)]\,\|\\
&< \|u-v\|_{N\lambda} L \sum_{k=0}^{\infty} \|S^{\smile}_{\alpha,\alpha}(k)\|,\end{aligned}$$

where $S^{\smile}_{\alpha,\alpha}(n) = \lambda^\wedge(-n) S_{\alpha,\alpha}(n)$. Then

$$\begin{aligned}&\|G(u) - G(v)\|_{N\lambda}\\&\quad = \max_{n\in[0,N]} \|\lambda^\wedge(-n) \sum_{j=-\infty}^{n} S_{\alpha,\alpha}(n-1-j)\,[f(j,u(j)) - f(j,v(j))]\,\|\\&\quad \le L\|u-v\|_{N\lambda}\|S^{\smile}_{\alpha,\alpha}\|_1 < \|u-v\|_{N\lambda},\end{aligned}$$

where by Theorem 3.1, Remark 3.3 (iv), and (2.7), we have

$$\begin{aligned}\|S^{\sim}_{\alpha,\alpha}\|_1 &\le \sum_{n=0}^{\infty} |\lambda|^{-n/N} \sum_{j=0}^{\infty} \varphi_{\alpha,0}(n,j) r_A^{j+1} = \sum_{n=0}^{\infty} |\lambda|^{-n/N} \mathcal{E}_{\alpha,\alpha}\left(1 - \frac{1}{r_A}, n\right)\\
&= \frac{|\lambda|^{\alpha/N}}{(|\lambda|^{1/N} - 1)^\alpha - (1 - \frac{1}{r_A})|\lambda|^{\alpha/N}} = \frac{1}{\left(1 - \frac{1}{|\lambda|^{1/N}}\right)^\alpha + \left(\frac{1}{r_A} - 1\right)}.\end{aligned}$$

Therefore, the conclusion follows from H_3. For the above, it follows that there exists a unique function $u \in \mathbb{P}_{N\lambda}(\mathbb{Z}, X)$, such that $Gu = u$. Hence, u is the unique (N,λ)-periodic mild solution of equation (4.4). □

Remark 4.6. Regarding condition H_3, we observe that it is enough to have the weaker condition $L\|S^{\smile}_{\alpha,\alpha}\|_1 < 1$ where $S^{\sim}_{\alpha,\alpha}(n):=\lambda^\wedge(-n)S_{\alpha,\alpha}(n)$ and $\{S_{\alpha,\alpha}(n)\}_{n\in\mathbb{N}_0}$ is the (α,α)-resolvent sequence generated by A.

We finally finish with an application of the main result presented in this paper.

Example. Let $0<\alpha<1$ and $|\lambda|\geq 1$. We consider the following fractional difference-differential equation in $X=L^2(0,1)$:

$$\begin{cases} \Delta^\alpha u(n,x) = \frac{\partial^2}{\partial x^2}u(n+1,x) + g(n,x)\cos(h(n,x)u(n,x)), & n\in\mathbb{Z},\ x\in(0,1), \\ u(n,0)=u(n,1)=0, \end{cases} \tag{4.7}$$

where $g\in\mathbb{P}_{N\lambda}(\mathbb{Z},L^2(0,1))$, $h\in\mathbb{P}_{N\frac{1}{\lambda}}(\mathbb{Z},L^2(0,1))$ and

$$\max_{n\in[0,N]}\|g(n)h(n)\|_{L^2} < \left(1-|\lambda|^{-1/N}\right)^\alpha + \left(\left(\sum_{m=1}^\infty \frac{1}{(1+(m\pi)^2)^2}\right)^{-1/2} - 1\right). \tag{4.8}$$

We define

$$\begin{aligned} D(A) &= \{f\in L^2(0,1) : f''\in L^2(0,1), f(0)=f(1)=0\}, \\ Af &= f'', \quad \forall f\in D(A). \end{aligned}$$

Then, (4.7) can be written in the abstract setting (4.4). It is well known that A is the generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ on $L^2(0,1)$ (see [25]) which is given by

$$T(t)f = \sum_{j=0}^\infty e^{-j^2\pi^2 t}\langle f,\mathbf{e}_j\rangle \mathbf{e}_j,$$

where $\{\mathbf{e}_j\}$ is the standard basis in $L^2(0,1)$. Moreover, we can represent the generator A as

$$Af = -\sum_{m=1}^\infty (m\pi)^2\langle f,\mathbf{e}_m\rangle\mathbf{e}_m, \qquad f\in D(A).$$

Then, for each $f\in L^2(0,1)$, we have $1\in\rho(A)$ and

$$\|(I-A)^{-1}f\|_{L^2}^2 = \sum_{m=1}^\infty \frac{1}{(1+(m\pi)^2)^2}|\langle f,\mathbf{e}_m\rangle|^2.$$

Note that

$$\begin{aligned} r_A := \sup_{\|f\|_{L^2}=1}\|(I-A)^{-1}f\|_{L^2} &= \left(\sum_{m=1}^\infty \frac{1}{(1+(m\pi)^2)^2}\right)^{1/2} \\ &\leq \left(\frac{1}{\pi^4}\sum_{m=1}^\infty \frac{1}{m^4}\right)^{1/2} = \frac{1}{\sqrt{90}} < 1, \end{aligned}$$

where we have used the formula [29, P. 651] in the last equality. Then, the condition (4.6) is satisfied. Now, we shall verify all the hypotheses in Theorem 4.5. Indeed, the sequence $f(n,\xi):=g(n)\cos(h(n)\xi)$, $\xi \in L^2(0,1)$, satisfies

$$\begin{aligned} f(n+N,\lambda\xi) &= g(n+N)\cos(h(n+N)\lambda\xi) = \lambda g(n)\cos\left(\frac{1}{\lambda}h(n)\lambda\xi\right) \\ &= \lambda g(n)\cos(h(n)\xi) = \lambda f(n,\xi), \end{aligned}$$

and

$$\|f(n,\xi)-f(n,\psi)\|_{L^2} \le \|g(n)h(n)\|_{L^2}\|\xi-\psi\|_{L^2} \le L\|\xi-\psi\|_{L^2},$$

where

$$L:=\max_{n\in[0,N]}\|g(n)h(n)\|_{L^2}.$$

From Eq. (4.8) and the fact that $r_A<1$, we obtain that

$$L<\left(1-\frac{1}{|\lambda|^{1/N}}\right)^{\alpha}+\left(\frac{1}{r_A}-1\right),$$

satisfying H_3. Thus, we have checked all the hypotheses of Theorem 4.5. Hence, Eq. (4.7) has a unique (N,λ)-periodic mild solution.

Finally, observe that in case $|\lambda|=1$, we have that

$$L:=\max_{n\in[0,N]}\|g(n)h(n)\|_{L^2}<\frac{1}{r_A}-1.$$

and therefore condition H_3 independent of α. This happens precisely in the standard cases of discrete periodic, discrete anti-periodic, and discrete Bloch-periodic functions.

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